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Bayesian Predictive Intervals Based on Type-II Hybrid Censored Data

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Abstract: Prediction of future events on the basis of the past and present knowledge is a fundamental problem of statistics, in this paper we discuss the one- and two- sample Bayesian prediction problem from generalized linear exponential distribution based on Type- II hybrid censored data. Fort his problem, the Gibbs sampling procedure and Lindley approximation are used to approximate the Bayesian predictive survival function and several hyper parameters are used to show the sensitivity of Bayesian Predictive Intervals with respect to these hyper parameters. Finally, some numerical result are presented.
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1. Introduction

Experiments often end before all test units have failed because of cost and time considerations. In such experiments failure information is available only on a part of the sample, the data are said to be censored data. Type-I and Type-II censoring schemes are the two most common censoring schemes. A combination of Type-I and Type-II censoring schemes, known as hybrid censoring scheme. It has been discussed to overcome the disadvantages of those two types separately. This scheme was first introduced by [10, 11], and it has been discussed extensively in the reliability literature. In the Type-I hybrid censoring scheme, the experiment is terminated as soon as a pre-specified number r out of n items has failed or a pre-fixed time x_0 on test has been reached. This censoring scheme was introduced by [10]. In contrast, in the Type-II hybrid censoring scheme, the life-testing experiment gets terminated whenever the later of the two stopping rules is reached; see [6]. Hybrid censored lifetime data have been discussed by several authors, including [14], [13], [16], [22], [5], [17], [21], [15], [25], [26] and [24]. One of the drawbacks of Type-I hybrid censoring scheme is that there may be very few failures occurring up to the pre-fixed time *T*. Because of this, [6] proposed a new hybrid censoring scheme known as Type-II hybrid censoring scheme in which the life testing experiment terminated whenever the later of the two stopping rules is reached.

Prediction of future events based on past and present knowledge is a fundamental statistical problem which arises naturally in many contexts. As in the case of estimation, a predictor can be either a predictor of a point or an interval. Several researches have considered Bayesian prediction for future observation based on Type-I censored data [1, 3]. Bayesian prediction bounds for future observation based on Type-II censored data have been discussed by several authors, including [8], [20] and [23]. [7] discussed the two - sample Bayesian prediction of the future life time of an item based on Type-I hybrid censored data from an exponential distribution. [9] developed the classical prediction intervals for future failures in the case of exponential distribution under Type-I hybrid censoring. Recently, [4] considered a general form for the inverted linear exponential distribution and a general conjugate prior and developed a general procedure for determining the one- and two – sample Bayesian prediction for future lifetimes based on a Type-II hybrid censored data. Prediction based on hybrid censored data also been discussed by several authors, see [26].

The generalized linear exponential distribution was originally proposed by [19], this distribution can be used for modeling bathtub, increasing and decreasing hazard rate behavior. This distribution is important because it contains some widely known distributions like exponential distribution, Rayleigh distribution, the linear exponential distribution and the Weibull distribution.

The generalized linear exponential distribution with three parameter α,θ,λ) distribution, has a probability density function (pdf) and cumulative distribution function (cdf) respectively.

$$f(x_i) = \alpha\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}} \quad (1)$$
$$\alpha,\theta > 0\ and \lambda \geq 0,$$

and

$$F(x_i) = 1 - e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}},\ \alpha,\theta > 0\ and\lambda \geq 0, \quad (2)$$

In this paper, we study the problem of prediction of the GLE (α,θ,λ) distribution based on Type-II hybrid censored data. For purposes of computations, the Gibbs sampling procedure and Lindley approximation are used to approximate the Bayesian predictive survival function and Lindley's approximation based on informative and non informative sample. In Section 2.2, one-sample Bayesian predictive survival function from GLE (α,θ,λ) distribution based on Type-II hybrid censored data are proposed. The two-sample Bayesian predictive survival function from GLE (α,θ,λ) distribution based on Type-II hybrid censored data is introduced in Section 2.3. Analysis of real data set and simulation are presented in Section 2.4 to illustrate the results.

2. Bayesian Prediction Intervals

The Bayes prediction of the future samples based on current sample, known as the informative sample, is an important problem in statistics. [2], proposed two main types of prediction problem, namely One-sample prediction and two sample prediction.

Suppose that $X_{1:n} < X_{2:n} <\ldots < X_{n:n}$ is a hybrid censored sample of size n drawn from a continuous distribution function $F(x)$ with density function $f(x)$. Let d denote the number of $X'_{i:n}s$ that are at most T. There fore, under the Type-II hybrid censoring scheme we have one of the two following types of observations:

Case I. $X_{1:n} < X_{2:n} < \ldots < X_{r:n}$ if $X_{r:n} > T$.
Case II. $X_{1:n} < X_{2:n} < \ldots < X_{d:n}$ if $TX_{r:n}$ with $r \leq d \leq n$.

Thus, the likelihood function of the Type-II hybrid censored sample is as follows:

Case I.

$$L(\alpha,\theta,\lambda) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i)\,[1-F(x_R)]^{n-r}, \quad (3)$$

where case II,

$$L(\alpha,\theta,\lambda) = \frac{n!}{(n-d)!}\prod_{i=1}^{d} f(x_i)\,[1-F(T)]^{n-d}. \quad (4)$$

Let $r < s \leq n$, then the conditional density function of $X_{s:n}$, given the Type-II hybrid censored sample, is as follows:

Case I.

$$f(x_s \mid x) = \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{(F(x_s)-F(x_r))^{s-r-1}(1-F(x_s))^{n-s}f(x_s)}{(1-F(x_r))^{n-r}}, \quad (5)$$

where $r < s \leq$, $x = (x_1, \ldots, x_r)$ and $x_s > x_r$.

Case II.

$$f(x_s \mid x) = \sum_{d=r}^{s-1} \frac{(n-d)!\phi_d(T)}{(s-d-1)!(n-s)!} \frac{(F(x_s)-F(T))^{s-d-1}(1-F(x_s))^{n-s}f(x_s)}{(1-F(T))^{n-d}} \tag{6}$$

where $x = (x_1, ..., x_d)$, $x_s > T$ and $\phi_d(T) = \frac{P(D=d)}{\sum_{j=r}^{s-1} P(D=j)}$.

Let $Y_{1:m} \leq Y_{2:m} \leq ... \leq Y_{m:m}$ be order statistics from a future random sample of size m from the same population. Then the marginal density function of the sth order statistics $Y_{s:m}$ is given by

$$f_{Y_{s:m}}(y_s) = \frac{m!}{(s-1)!(m-s)!}(F(y_s))^{s-1}(1-F(y_s))^{m-s}f(y_s), \tag{7}$$

where $1 \leq s \leq m$.

Since the survival function $\bar{F}(x|\theta) = 1 - F(x|\theta)$ corresponding to any cumulative distribution function $F(x|\theta), \theta \in \Theta$, where Θ may be a vector of parameters of the distribution can be written as

$$\bar{F}(x|\theta) = \exp[-\lambda(x,\theta)], \tag{8}$$

where $\lambda(x,\theta) = -\ln \bar{F}(x|\theta)$,we shall consider the underling population survival function to be given by (8).

The probability density function corresponding to (8) is given by

$$f(x|\theta) = \acute{\lambda}(x,\theta)\exp[-\lambda(x,\theta)], \tag{9}$$

where $\acute{\lambda}(x,\theta)$ is the derivative of $\lambda(x,\theta)$ with respect to x.

For the Bayesian prediction setup, we need a suitable prior parameter distribution. Suggested by [2], that is given by

$$\pi(\theta,\delta) \propto C(\theta,\delta)\exp[-D(\theta,\delta)], \tag{10}$$

where $\theta \in \Theta$ and δ is the vector of prior parameters.

2.1. One-sample Bayesian prediction

We discuss the Bayesian prediction for the GLE (α,θ,λ) where the three parameters α,θ and λ are unknown.

Suppose that $X_{1:n} < X_{2:n} < ... < X_{n:n}$ is a hybrid censored sample of size n drawn from a population with pdf given by 1, we can rewrite the pdf as $f(x;\alpha,\theta,\lambda) = H^0(x;\alpha,\theta,\lambda)e^{-H(x;\alpha,\theta,\lambda)}$, $\alpha,\theta > 0$ *and* $\lambda \geq 0$, and cumulative distribution function (cdf)

$$f(x;\alpha,\theta,\lambda) = H^{'}(x;\alpha,\theta,\lambda)e^{-H(x;\alpha,\theta,\lambda)}, \alpha,\theta > 0 \; and \lambda \geq 0,$$

and cumulative distribution function (cdf)

$$F(x,\alpha,\theta,\lambda) = 1 - e^{-H(x;\alpha,\theta,\lambda)}, \alpha,\theta > 0 \; and \lambda \geq 0, \tag{11}$$

where

$$H(x;\alpha,\theta,\lambda) = \left(\lambda x + \frac{\theta}{2}x^2\right)^{\alpha},$$

and $H^{'}(x;\alpha,\theta,\lambda)$ is the derivative of $H(x;\alpha,\theta,\lambda)$ with respect to x, so,

$$\acute{H}(x;\alpha,\theta,\lambda) = \alpha\left(\lambda x + \frac{\theta}{2}x^2\right)^{\alpha-1}(\lambda + \theta x).$$

Upon using (8) and (9) in (3) and (4), we obtain the likelihood function as follows:

Case I

$$L_1(\alpha,\theta,\lambda,\underline{x}) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} H'(x;\alpha,\theta,\lambda)\right) e^{-\sum_{i=1}^{r}[H(x_i;\alpha,\theta,\lambda)+(n-r)H(x_r;\alpha,\theta,\lambda)]}, \tag{12}$$

where $\underline{x}=(x_1,...,x_r), x_1<...<x_r$ and $x_r>T$,

Case II

$$L_2(\alpha,\theta,\lambda,\underline{x}) = \frac{n!}{(n-d)!}\left(\prod_{i=1}^{d} H'(x;\alpha,\theta,\lambda)\right) e^{-\sum_{i=1}^{d}[H(x_i;\alpha,\theta,\lambda)+(n-d)H(T;\alpha,\theta,\lambda)]}, \tag{13}$$

where $\underline{x}=(x_1,...,x_r), x_1<...<x_d\le T<x_r$.

Similarly, upon substituting (8) and (9) in (5)and (6), we obtain the conditional density function of $X_{s:n}$, given the Type-II hybrid censored sample, as follows :

Case I

$$f(x_s\mid x)=\sum_{\omega=0}^{s-r-1} C_1 H'(x_s;\alpha,\theta,\lambda)g(x_s,x_r;\alpha,\theta,\lambda) \tag{14}$$

where $\underline{x}=(x_{1,}...,x_r)$, $x_s>x_r$., $C_1=\frac{(-1)^{\omega}s-r-1\omega(n-r)!}{(s-r-1)!(n-s)!}$ and

$$g(x_s,x_r;\alpha,\theta,\lambda)=e^{-(n-s+\omega+1)[H(x_s;\alpha,\theta,\lambda)-H(x_r;\alpha,\theta,\lambda)]}, \tag{15}$$

Case II

$$f(x_s\mid x)=\sum_{d=r}^{s-1}\sum_{\omega=0}^{s-d-1} C_2\phi_d(T,\alpha,\theta,\lambda)H'(x_s;\alpha,\theta,\lambda)g(x_s,T;\alpha,\theta,\lambda), \tag{16}$$

where $\underline{x}=(x_{1,}...,x_d)$, $x_s>T.$, $C_2=\frac{(-1)^{\omega}s-d-1\omega(n-d)!}{(s-d-1)!(n-s)!}$ and

$$\phi_d(T,\alpha,\theta,\lambda)=\frac{nd\exp[-(n-d)H(T;\alpha,\theta,\lambda)]\{1-\exp[-H(T;\alpha,\theta,\lambda)]\}^d}{\sum_{j=r}^{s-1} nj\exp[-(n-j)H(T;\alpha,\theta,\lambda)]\{1-\exp[-H(T;\alpha,\theta,\lambda)]\}^j}. \tag{17}$$

For the case when the parameters α,θ and λ are unknown, we use the following independent prior:

$\pi_1(\alpha)\propto\alpha^{w_2-1}e^{-\alpha w_1}$, $w_1>0, w_2>0$ $\alpha>0$,

$$\pi_2(\theta)\propto\theta^{w_4-1}e^{-\theta w_3} \quad ,w_3>0, w_4>0,\ \ \theta>0, \tag{18}$$

and $\pi_3(\lambda)\propto\lambda^{w_6-1}e^{-\lambda w_5}$, $w_6>0, w_5>0,\ \lambda\ge 0$, where $w_1, w_2, w_3, w_4, w_5, w_6$ are chosen to reflect prior knowledge about α, θ and λ.When $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$, there are non-informative priors of α, θ and λ.

From Equations (10) and (2.1) , the joint prior density function of α, θ and λ is of the form

$$\pi(\alpha, \theta, \lambda) \propto \alpha^{w_2-1} e^{-\alpha w_1} \theta^{w_4-1} e^{-\theta w_3} \lambda^{w_6-1} e^{-\lambda w_5} \quad , \alpha, \theta > 0, \ \lambda \geq 0,$$

$$w_1, w_2, w_3, w_4, w_5, w_6 > 0.$$

So, we have

$$C(\alpha, \theta, \lambda, \delta) = \alpha^{w_2-1} \theta^{w_4-1} \lambda^{w_6-1} and \ \ D(\alpha, \theta, \lambda, \delta) = \alpha w_1 + \theta w_3 + \lambda w_5, \quad (19)$$

where $\delta = (\ w_1, w_2, w_3, w_4, w_5, w_6)$.

Since the posterior density function is given by

$$\pi^*(\alpha, \theta, \lambda \mid x) = \frac{1}{k} \pi(\alpha, \theta, \lambda, \delta) \quad L(\alpha, \theta, \lambda, x), \quad (20)$$

where

$$k = \int_{\theta \in \Theta} \pi(\alpha, \theta, \lambda, \delta) \quad L(\alpha, \theta, \lambda, x)\, d\theta, \quad (21)$$

Using Equations (10), (12), (13) and (20) we can write the posterior density function as:

Case I.

$$\pi_1^*(\alpha, \theta, \lambda \mid x) = \frac{1}{k_1} \eta_1(\alpha, \theta, \lambda, x) \quad \exp[-\zeta 1(\alpha, \theta, \lambda, x)], \quad (22)$$

where

$$\eta_1(\alpha, \theta, \lambda, x) = C(\alpha, \theta, \lambda, \delta) \prod_{i=1}^{r} H'(x_i; \alpha, \theta, \lambda),$$

$$\zeta 1(\alpha, \theta, \lambda, x) = \sum_{i=1}^{r} H(x_i; \alpha, \theta, \lambda) + (n-r) H(x_r; \alpha, \theta, \lambda) + D(\alpha, \theta, \lambda, \delta).$$

and

$$k_1 = \int_0^\infty \int_0^\infty \int_0^\infty \eta_1(\alpha, \theta, \lambda, x) \quad \exp[-\zeta 1(\alpha, \theta, \lambda, x)] d\alpha d\theta d\lambda,$$

Case II.

$$\pi_2^*(\alpha, \theta, \lambda \mid x) = \frac{1}{k_2} \eta_2(\alpha, \theta, \lambda, x) \quad \exp[-\zeta_2(\alpha, \theta, \lambda, x)], \quad (23)$$

where

$$\eta_2(\alpha,\theta,\lambda,x) = C(\alpha,\theta,\lambda,\delta)\prod_{i=1}^{d} H^{'}(x_i;\alpha,\theta,\lambda),$$

$$\zeta_2(\alpha,\theta,\lambda,x) = \sum_{i=1}^{d} H(x_i;\alpha,\theta,\lambda) + (n-d)H(T;\alpha,\theta,\lambda) + D(\alpha,\theta,\lambda,\delta),$$

and

$$k_2 = \int_0^\infty\int_0^\infty\int_0^\infty \eta_2(\alpha,\theta,\lambda,x)\ \exp[-\zeta_2(\alpha,\theta,\lambda,x)],d\alpha d\theta d\lambda,$$

The Bayesian predictive density function of $x_{s:n}$ is given by

$$p(x_s \mid x) = \int_{\theta\in\Theta} f(x_s \mid x)\pi^*(\alpha,\theta,\lambda \mid x)\,d\alpha d\theta d\lambda. \tag{24}$$

Upon using 14, 16, 22, 23 in 24, we obtain the predictive density function of $x_{s:n}$ as follows:

Case I.

$$p(x_s \mid x) = \sum_{\omega=0}^{s-r-1} C_1 \int_0^\infty\int_0^\infty\int_0^\infty H^{'}(x_s;\alpha,\theta,\lambda)g(x_s,x_r;\alpha,\theta,\lambda)\pi_1^*(\alpha,\theta,\lambda \mid x)\,d\alpha d\theta d\lambda. \tag{25}$$

where $x=(x_{1,}...,x_r)$, $x_s > x_r$.,

Case II.

$$p(x_s \mid x) = \sum_{d=r}^{s-1}\sum_{\omega=0}^{s-d-1} C_2 \tag{26}$$

$$\times \int_0^\infty\int_0^\infty\int_0^\infty \phi_d(T,\alpha,\theta,\lambda)H^{'}(x_s;\alpha,\theta,\lambda)g(x_s,T;\alpha,\theta,\lambda)\pi_2^*(\alpha,\theta,\lambda \mid x)\,d\alpha d\theta d\lambda.$$

where $x=(x_{1,}...,x_d)$, $x_s > T$.

So, the predictive survival function of $x_{s:n}$ is obtained as follows:

Case I.

$$P(x_s > t \mid x) = \sum_{\omega=0}^{s-r-1} C_1 \tag{27}$$

$$\times \int_t^\infty\int_0^\infty\int_0^\infty\int_0^\infty H^{'}(x_s;\alpha,\theta,\lambda)g(x_s,x_r;\alpha,\theta,\lambda)\pi_1^*(\alpha,\theta,\lambda \mid x)\,d\alpha d\theta d\lambda dx_s$$

$$=\sum_{\omega=0}^{s-r-1}\frac{C_1}{n-s+\omega+1}\int_0^\infty\int_0^\infty\int_0^\infty g(t,x_r;\alpha,\theta,\lambda)\pi_1^*(\alpha,\theta,\lambda \mid x)\,d\alpha d\theta d\lambda.$$

Case II.

$$P(x_s > t \mid x) = \sum_{d=r}^{s-1} \sum_{\omega=0}^{s-d-1} C_2 \tag{28}$$

$$\times \int_t^\infty \int_0^\infty \int_0^\infty \int_0^\infty \phi_d(T,\alpha,\theta,\lambda) H'(x_s;\alpha,\theta,\lambda) g(x_s,T;\alpha,\theta,\lambda) \pi_2^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda dx_s.$$

$$= \sum_{d=r}^{s-1} \sum_{\omega=0}^{s-d-1} \frac{C_2}{n-s+\omega+1} \int_0^\infty \int_0^\infty \int_0^\infty \phi_d(T,\alpha,\theta,\lambda) g(t,T;\alpha,\theta,\lambda) \pi_2^*(\alpha,\theta,\lambda \mid x)\, d\alpha dt$$

Then, the $100\tau\%$ Bayesian prediction interval (L, U) for $X_{s:n}$ based on the Type-II hybrid censored sample is obtained by solving the following two equations:

$$P(x_{s:n} > L \mid x) = \frac{1+\tau}{2} \tag{29}$$

and

$$P(x_{s:n} > U \mid x) = \frac{1-\tau}{2} \tag{30}$$

It does not seem to be possible to compute the probability in (27) and (28) analytically. We therefore use Markov Chain Monte Carlo (MCMC) technique and Lindley approximation are used for constructing the Bayesian prediction intervals (BPIs).

2.1.1. MCMC Method

Using MCMC to generate α, θ and λ from the posterior density function, so the marginal posterior density of α, is proportional to

$$\pi_{11}^*(\alpha \mid \theta, \lambda,\ x) \propto \alpha^{w_2+r-1} e^{-\alpha w_1} \tag{31}$$

$$\times \prod_{i=1}^{R} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda + \theta x_i)\, e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^\alpha} e^{-\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha(n-R)}}$$

Similarly, the posterior conditional distribution for θ and λ are respectively

$$\pi_{22}^*(\theta \mid \alpha, \lambda,\ x) \propto \theta^{w_4+r-1} e^{-\theta w_3} \tag{32}$$

$$\times \prod_{i=1}^{R} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda + \theta x_i)\, e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^\alpha} e^{-\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha(n-R)}}$$

and

$$\pi_{33}^*(\lambda \mid \alpha, \theta,\ x) \propto \theta^{w_0+r-1} e^{-\theta w_5} \tag{33}$$

$$\times \prod_{i=1}^{R} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda + \theta x_i)\, e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^\alpha} e^{-\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha(n-R)}}$$

Where

$$R = \begin{cases} r & forcaseI \\ d & forcaseII \end{cases},$$

and

$$c = \begin{cases} x_{R:n} & forcaseI \\ T & forcaseII \end{cases}.$$

To compute $\int_0^\infty \int_0^\infty \int_0^\infty f(x_s \mid x)\pi^*(\alpha,\theta,\lambda \mid x)\, d\alpha\ d\theta\ d\lambda$.by using the MCMC technique, we use the following procedure:

1) Set the initial values of α, θ and λ say (α_0, θ_0,λ_0).
2) Set j=1.
3) Using MH, generate α_1^j from $\pi_{11}^*(\alpha^{j-1} \mid \theta^{j-1},\lambda^{j-1},x)$ with normal distribution, N(α^{j-1},KαVα).
4) Using MH, generate θ_1^j from $\pi_{22}^*(\theta^{j-1} \mid \alpha^{j-1},\lambda^{j-1},x)$ with normal distribution, N(θ^{j-1},$K_\theta V_\theta$).
5) Using MH, generate λ_1^j from $\pi_{33}^*(\lambda^{j-1} \mid \alpha^{j-1},\theta^{j-1},x)$ with normal distribution, N(λ^{j-1},$K_\lambda V_\theta$),
where Kα,K_θ and K_λ are scaling factor and Vα , V_θ and V_θ are variances-co variances matrix.
6) Set j=j+1.
7) Repeat steps from 1 to N times.
8) The approximate value of $\int_0^\infty \int_0^\infty \int_0^\infty f(x_s \mid x)\pi^*(\alpha,\theta,\lambda \mid x)\, d\alpha\ d\theta\ d\lambda$ is then obtained as

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x_s \mid x)\pi^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda = \frac{\sum_{i=1}^N\ f(\alpha_i,\theta_i,\lambda_i)}{N},$$

where

Case I

$$f(\alpha,\theta,\lambda) = \sum_{\omega=0}^{s-r-1} \frac{(-1)^\omega s-r-1\omega(n-r)!}{(n-s+\omega+1)(s-r-1)!(n-s)!} e^{-(n-s+\omega+1)\left[\left(\lambda t+\frac{\theta}{2}t^2\right)^\alpha-\left(\lambda x_r+\frac{\theta}{2}x_r^2\right)^\alpha\right]},$$

Case II

$$f(\alpha,\theta,\lambda) = \sum_{d=r}^{s-1}\sum_{\omega=0}^{s-d-1} \frac{(-1)^\omega s-d-1\omega(n-d)!}{(n-s+\omega+1)(s-d-1)!(n-s)!}$$
$$\times \frac{nd \exp\left[-(n-d)\left(\lambda T+\frac{\theta}{2}T^2\right)^\alpha\right]^{n-d}\left\{1-\exp\left[-\left(\lambda T+\frac{\theta}{2}T^2\right)^\alpha\right]\right\}^d}{\sum_{j=r}^{s-1} nj \exp\left[-(n-j)\left(\lambda T+\frac{\theta}{2}T^2\right)^\alpha\right]^{n-j}\left\{1-\exp\left[-\left(\lambda T+\frac{\theta}{2}T^2\right)^\alpha\right]\right\}^j}$$
$$\times e^{-(n-s+\omega+1)\left[\left(\lambda t+\frac{\theta}{2}t^2\right)^\alpha-\left(\lambda T+\frac{\theta}{2}T^2\right)^\alpha\right]}.$$

2.1.2. Lindely Approximation Method

We derive the BPIs based on Lindley approximation, which introduced by [18] can approximate the Bayes estimators into a form containing no integral. For our estimation problem we describe this method below. As noticed the Bayesian estimates include the ratio of two integrals, we consider $I(x)$ defined as

$$I(x) = E[u(\gamma_1, \gamma_2, \gamma_3)] = \frac{\int\limits_{\gamma_3}\int\limits_{\gamma_2}\int\limits_{\gamma_1} u(\gamma_1, \gamma_2, \gamma_3)\, e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3}{\int\limits_{\gamma_3}\int\limits_{\gamma_2}\int\limits_{\gamma_1} e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3},$$

where $u(\gamma_1, \gamma_2, \gamma_3)$ is a function of γ_1, γ_2 or γ_3 only.
$L(\gamma_1, \gamma_2, \gamma_3)$ is log of likelihood function.
$\rho(\gamma_1, \gamma_2, \gamma_3)$ is log joint prior of γ_1, γ_2 and γ_3.
Utilizing the Lindley's method $I(x)$ can be approximated as

$$I(x) = u(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) +$$

$$\frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) \quad (34)$$
$$+C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})],$$

where
$\hat{\gamma}_1, \hat{\gamma}_2$ and $\hat{\gamma}_3$ are the MLE of of γ_1, γ_2 and γ_3 respectively.

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \; i = 1, 2, 3,$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},$$

$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}),$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331},$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332},$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333},$$

and subscripts $1, 2, 3$ on the right-hand sides refer to γ_1, γ_2 , γ_3 respectively and

$$\rho_i = \frac{\partial\rho}{\partial\gamma i}, \quad u_i = \frac{\partial u(\gamma_1, \gamma_2, \gamma_3)}{\partial\gamma i}, \; i = 1, 2, 3, \; u_{ij} = \frac{\partial^2 u(\gamma_1, \gamma_2, \gamma_3)}{\partial\gamma_i\partial\gamma_j}, \; i, j = 1, 2, 3,$$

.

$$L_{ij}=\frac{\partial^2 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma i\partial\gamma_j},\ i,j=1,2,3\ ,\quad L_{ijk}=\frac{\partial^3 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma_i\partial\gamma_j\partial\gamma_k},\ i,j,k=1,2,3,$$

and σ_{ij} is the $(i,j)-th$ element of the inverse of the matrix $\{L_{ij}\}$, all evaluated at the MLE of parameters.With the above defined expressions, we can obtain the values of the Bayes estimates of various parameters .

Replacing $u(\gamma_1,\gamma_2,\gamma_3)$ by

Case I

$$f(\alpha,\theta,\lambda)=\sum_{\omega=0}^{s-r-1}\frac{(-1)^{\omega}s-r-1\omega(n-r)!}{(s-r-1)!(n-s)!} \tag{35}$$

$$\times\alpha(\lambda+\theta x_s)\left(\lambda x_s+\tfrac{\theta}{2}x_s^2\right)^{\alpha-1}e^{-(n-s+\omega+1)\left[\left(\lambda x_s+\frac{\theta}{2}x_s^2\right)^{\alpha}-\left(\lambda x_r+\frac{\theta}{2}x_r^2\right)^{\alpha}\right]}$$

Case II

$$f(\alpha,\theta,\lambda)=\sum_{d=r}^{s-1}\sum_{\omega=0}^{s-d-1}\frac{(-1)^{\omega}s-d-1\omega(n-d)!\phi_d(T)}{(s-d-1)!(n-s)!} \tag{36}$$

$$\times\alpha(\lambda+\theta x_s)\left(\lambda x_s+\tfrac{\theta}{2}x_s^2\right)^{\alpha-1}e^{-(n-s+\omega+1)\left[\left(\lambda x_s+\frac{\theta}{2}x_s^2\right)^{\alpha}-\left(\lambda T+\frac{\theta}{2}T^2\right)^{\alpha}\right]}$$

2.2. Two-sample Bayesian Prediction

Let $x_{m:n}$ $Y_{1:m}$,$Y_{2:m}$ ... $Y_{m:m}$ be the order statistics (OS) from a future random sample of size m from the same population. Then the marginal density function of the sth OS $Y_{s:m}$ is given by

$$f_{Y_{m:m}}(y_s)=\frac{m!}{(s-1)!(m-s)!}(F(y_s))^{s-1}(1-F(y_s))^{m-s}f(y_s) \tag{37}$$

where $1\le s\le m$. $m\le d<n$ Upon substituting (1), (2) in 37, we obtain

$$f(y_s\mid x)=\sum_{\omega=0}^{s-1}C_3H'(y_s;\alpha,\theta,\lambda)e^{-(m-s+\omega+1)H(y_s;\alpha,\theta,\lambda)} \tag{38}$$

where $C_3=\frac{m!}{(s-1)!(m-s)!}s-1\omega(-1)^{\omega}$

Since the Bayesian predictive density function of $Y_{s:m}$ is given by

$$p(y_s\mid x)=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}f(y_s\mid x)\pi^*(\alpha,\theta,\lambda\mid x)\,d\alpha d\theta d\lambda. \tag{39}$$

upon substituting (22), (23), and (38) in (39), we obtain the Bayesian predictive density function of $Y_{s:m}$ as follows:

Case I

$$p(y_s \mid x) = \sum_{\omega=0}^{s-1} C_3 \int_0^\infty \int_0^\infty \int_0^\infty H^{'}(y_s;\alpha,\theta,\lambda) e^{-(m-s+\omega+1)H(y_s;\alpha,\theta,\lambda)} \pi_1^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda, \tag{40}$$

Case II

$$p(y_s \mid x) = \sum_{\omega=0}^{s-1} C_3 \int_0^\infty \int_0^\infty \int_0^\infty H^{'}(y_s;\alpha,\theta,\lambda) e^{-(m-s+\omega+1)H(y_s;\alpha,\theta,\lambda)} \pi_2^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda. \tag{41}$$

From the above two equations, we obtain the predictive survival function of $Y_{s:m}$ as follows:

Case I

$$P(y_s > t \mid x) = \sum_{\omega=0}^{s-1} C_3 \int_t^\infty \int_0^\infty \int_0^\infty \int_0^\infty H^{'}(y_s;\alpha,\theta,\lambda) e^{-(m-s+\omega+1)H(y_s;\alpha,\theta,\lambda)} \pi_1^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda dy_s$$

$$= \sum_{\omega=0}^{s-1} \frac{C_3}{m-s+\omega+1} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(m-s+\omega+1)H(t;\alpha,\theta,\lambda)} \pi_1^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda. \tag{42}$$

Case II

$$P(y_s > t \mid x) = \sum_{\omega=0}^{s-1} C_3 \int_t^\infty \int_0^\infty \int_0^\infty \int_0^\infty H^{'}(y_s;\alpha,\theta,\lambda) e^{-(m-s+\omega+1)H(y_s;\alpha,\theta,\lambda)} \pi_2^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda dy_s$$

$$= \sum_{\omega=0}^{s-1} \frac{C_3}{m-s+\omega+1} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(m-s+\omega+1)H(t;\alpha,\theta,\lambda)} \pi_2^*(\alpha,\theta,\lambda \mid x)\, d\alpha d\theta d\lambda. \tag{43}$$

Consequently, the 100 % Bayesian prediction interval (L, U) for $Y_{s:m}$ based on the Type-II hybrid censored sample is obtained by solving the following two equations:

$$P(y_s > L \mid x) = \frac{1+\tau}{2} \tag{44}$$

and

$$P(y_s > U \mid x) = \frac{1-\tau}{2} \tag{45}$$

As the same as previous subsection, we use the MCMC method and Lindley approximation to determine the BPIs with $f(\alpha,\theta,\lambda)$

$$f(\alpha,\theta,\lambda) = \sum_{\omega=0}^{s-1} \frac{(-1)^\omega s - 1\omega n!}{(n-s+\omega+1)(s-1)!(n-s)!} e^{-(n-s+\omega+1)\left[\left(\lambda t+\frac{\theta}{2}t^2\right)^\alpha - \left(\lambda x_r+\frac{\theta}{2}x_r^2\right)^\alpha\right]}.$$

3. Numerical Results

To illustrate the inferential procedures developed in the preceding subsections, we present a numerical study for the considered distribution. To illustrate the prediction results for the GLE (α,θ,λ) when three parameters α,θ and λ are unknown, we generated order statistics from a sample of size $n = 10$ from the GLE distribution. The generated order statistics from the GLE (with $\alpha = 0.8, \theta = 0.5$ and $\lambda = 0.7$) are as follows: 0.0107609,0.32776, 0.38692, 0.443063, 0.8135, 1.14747, 1.23987, 2.78955, 4.2239 and 4.24113.

We shall use these data to consider two different Type-II hybrid censoring schemes.

1- When $r = 3$ and $T = 0.1$. Since $x_{3:10} > T$, the testing would have terminated in this case at time $x_{3:10} = 0.38692$ and we have obtained the following data: 0.0107609,0.32776 and 0.38692.

2- When $r = 2$ and $T = 0.4$. Since $T > x_{2:10}$, the testing would have terminated in this case at T and we would have obtained the following data 0.0107609 and 0.32776.

We assume these data to have come from the GLE (α,θ,λ) distribution, where three parameters α,θ and λ are unknown. Based on the above two Type-II hybrid censoring schemes, we then used the results presented in subsection 2.1 to constructed 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:n}, s = 4,\ldots,10,$ from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}, s = 1,\ldots 10,$ from a future sample of size $m = 10$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyper parameters $(w_1, w_2, w_3, w_4, w_5, w_6)$,

Table 1: 95% one-sample Bayesian prediction bounds for $X_{s:n}$,s = 4,...,10, from GLE distribution case I

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	$LX{:}s$	$UX{:}s$	$LX{:}s$	$UX{:}s$
4	0.397364	1.62064	0.396074	2.55095
5	0.492372	2.26373	0.470125	3.70406
6	0.669455	2.91469	0.601575	4.90543
7	0.91536	3.66836	0.783274	6.34046
8	1.23131	4.59977	1.02698	8.27948
9	1.6589	5.9951	1.3695	11.428
10	2.31958	8.92986	1.93059	19.0825
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	$LX{:}s$	$UX{:}s$	$LX{:}s$	$UX{:}s$
4	0.397357	1.61926	0.390069	0.833344
5	0.492551	2.26903	0.420353	1.20995
6	0.670586	2.92268	0.481959	1.8919
7	0.914561	3.66212	0.574731	2.67448
8	1.23355	4.61224	0.705472	3.62296
9	1.65947	5.99891	0.893163	5.07495
10	2.31843	8.92279	1.19901	8.46856

Table 2: 95% one-sample Bayesian prediction bounds for X s:n,s = 4,...,10, from GLE distribution case II

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	$LX{:}s$	$UX{:}s$	$LX{:}s$	$UX{:}s$
4	0.436785	2.12081	0.418215	3.42356
5	0.529006	2.72182	0.438048	4.49073
6	0.729021	3.38946	0.494302	5.75187
7	1.01591	4.1901	0.646235	7.33423
8	1.3797	5.21255	0.929498	9.52947
9	1.8601	6.76462	1.34638	13.1652
10	2.59701	10.0753	2.00882	22.1862
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	$LX{:}s$	$UX{:}s$	$LX{:}s$	$UX{:}s$
4	0.436869	2.12554	0.404276	0.856248
5	0.529112	2.72351	0.406585	0.933095
6	0.72877	3.39145	0.411557	2.06392
7	1.01451	4.18123	0.425335	2.37685
8	1.38029	5.21616	0.471732	3.52741
9	1.86039	6.76039	0.618597	5.18567
10	2.59556	10.0627	0.958795	8.99028

1 and 2 presents the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:n}, s = 4,\ldots,10,$ for the choices of $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$ and $w_1 = 2, w_2 = 3, w_3 = 1, w_4 = 1, w_5 = 2, w_6 = 3$. The lower and upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}, s = 1,\ldots,10,$ from a future sample, are presented in Table 3 and Table 4.

3.1. Conclusion

The previous section presents the BPIs from GLE distribution under MCMC technique and Lindley approximation. For informative and non informative hyper parameters, the following are observed:

From Tales 1-4, it is clear that, the lower bounds are relative insensitive to the specification of the hyper parameters $(w_1, w_2, w_3, w_4, w_5, w_6)$ while the upper bounds are almost insensitive.

Moreover, Tables 1-4 show that the Bayesian prediction bounds based on Lindley approximation are more close to each other to the specification of the hyper parameters $(w_1, w_2, w_3, w_4, w_5, w_6)$ than the Bayesian prediction bounds based on MCMC method.

Table 3: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$, s = 1,...,10, from GLE distribution case I

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	$LY{:}s$	$UY{:}s$	$LY{:}s$	$UY{:}s$
1	0.00309423	0.332859	0.000134658	0.45472
2	0.0376391	0.468727	0.00571803	0.647135
3	0.0927334	0.56963	0.0262002	0.814096
4	0.155803	0.6789	0.0627917	0.978962
5	0.220756	0.781392	0.112227	1.15391
6	0.291511	0.899273	0.171592	1.35126
7	0.374941	1.0587	0.240165	1.58965
8	0.46276	1.22987	0.320203	1.9062
9	0.567634	1.46959	0.41903	2.39976
10	0.731292	2.00051	0.560121	3.50444
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	$LY{:}s$	$UY{:}s$	$LY{:}s$	$UY{:}s$
1	0.00307784	0.338412	0.0000918488	0.406021
2	0.0371989	0.456593	0.00412405	0.609823
3	0.0928777	0.569421	0.0199296	0.783569
4	0.155873	0.677953	0.0505335	0.953933
5	0.222739	0.790389	0.0951987	1.13408
6	0.294001	0.909901	0.152097	1.3369
7	0.371384	1.0477	0.220226	1.58164
8	0.46052	1.22385	0.30103	1.90649
9	0.567692	1.46972	401183	2.4131
10	0.724437	1.98156	0.544073	3.54779

Table 4: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$, s = 1,...,10, from GLE distribution case II

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	$LY{:}s$	$UY{:}s$	$LY{:}s$	$UY{:}s$
1	0.00420059	0.35538	0.00306869	0.335226
2	0.0450995	0.483088	0.0372651	0.461039
3	0.105978	0.592571	0.0928646	0.570238
4	0.172112	0.698454	0.155312	0.676455
5	0.241008	0.807449	0.22177	0.786561
6	0.312406	0.922276	0.292881	0.906965
7	0.391556	1.0587	0.370876	1.04663
8	0.481079	1.231	0.459979	1.2223
9	0.590443	1.47667	0.569073	1.47595
10	0.744267	1.96272	0.724437	1.98156
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	$LY{:}s$	$UY{:}s$	$LY{:}s$	$UY{:}s$
1	0.00421709	0.355654	0.000325946	0.405464
2	0.0451268	0.483628	0.00968543	0.586801
3	0.105854	0.591012	0.0380055	0.745661
4	0.171831	0.696556	0.0831989	0.901598
5	0.240017	0.804433	0.141067	1.06517
6	0.313786	0.926155	0.209573	1.247
7	0.392311	1.06226	0.289078	1.46291
8	0.481745	1.23304	0.382596	1.74381
9	0.591611	1.48079	0.4983	2.1709
10	0.74651	1.96842	0.663147	3.09749

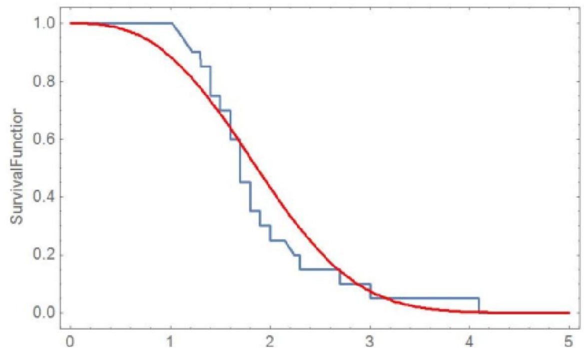

Fig 1. Empirical and fitted distribution function for completed data set.

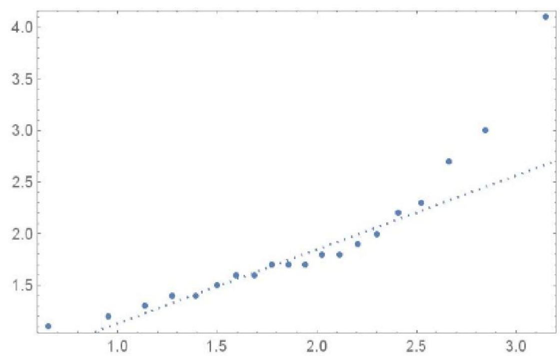

Fig 2. Q-Q plot compare data to a specific distribution.

4. Real Data Analysis

Purpose. The following data represents the relief times of twenty patients receiving an analgesic. This data set was taken from [12].

1.1, 1.2, 1.3, 1.4, 1.4, 1.5, 1.6, 1.6, 1.7, 1.7, 1.7, 1.8, 1.8, 1.9, 2., 2.2, 2.3, 2.7, 3., 4.1.

We have created two artificially hybrid censored data sets from the above data set, using the following censoring schemes:

Scheme I $r = 15,\ T = 1.5$. Since $x_{15,20} > T$, the life test would have terminated in this case at T, and we would have obtained the following data.

1.1, 1.2, 1.3, 1.4, 1.4, 1.5, 1.6, 1.6, 1.7, 1.7, 1.7, 1.8, 1.8, 1.9 and 2.

Scheme II $r = 14,\ T = 2$. Since $x_{14,20} < T$, the life test would have terminated in this case at $x_{14,20}$, and we would have obtained the following data.

1.1, 1.2, 1.3, 1.4, 1.4, 1.5, 1.6, 1.6, 1.7, 1.7, 1.7, 1.8, 1.8 and 1.9.

Before progressing, first we would like to check whether the GLED fit this data or not. The calculated value of the K-S test is 0.18497 for the GLE distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.29407 at $n = 20$. We have just plotted the empirical survival function and the fitted survival functions in FIGURE 1. Observe that the GLE distribution can be a good model fitting this data. FIGURE 2 shows that all points of a Q-Q plot are inside the unit square, so, it can be seen that the GLE distribution fits the data well.

In all the cases $\alpha = 1.9, \theta = 0.1$ and $\lambda = 0.3$. To construct 95% one sample BPIs for order statistics $X_{s:n}, s = 16,...,20,$ from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}, s = 1,...20,$ from a future sample of size $m = 20,$ the results in subsections 2.1.1, 2.1.2 and 2.2 are used. The corresponding results for one-sample and two-sample predictions, for the choices of the hyper parameters are presented in Tables 5 to Table 10, respectively.

Table 5: 95% one-sample Bayesian prediction bounds for X $_{s:n}$,s = 16,...,20, from GLE distribution case I

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	LX:s	UX:s	LX:s	UX:s
16	2.01247	2.86263	2.01382	3.04435
17	2.12139	3.15442	2.12697	3.38226
18	2.30241	3.41405	2.30651	3.68154
19	2.52317	3.7084	2.52146	4.01831
20	2.80246	4.16257	2.79226	4.54379
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	LX:s	UX:s	LX:s	UX:s
16	2.01244	2.86206	2.01055	5.0
17	2.12078	3.14996	2.0103	5.0
18	2.30225	3.41471	2.25477	3.57918
19	2.52298	3.70884	2.44975	3.91866
20	2.80118	4.16361	2.70488	4.44576

Table 6: 95% one-sample Bayesian prediction bounds for $X_{s:n}$,s = 16,...,20, from GLE distribution case II

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	LX:s	UX:s	LX:s	UX:s
16	2.01247	2.86263	2.05055	3.08882
17	2.12139	3.15442	2.22623	3.49737
18	2.30241	3.41405	2.40504	3.75656
19	2.52317	3.7084	2.60671	4.06739
20	2.80246	4.16257	2.85967	4.57226
MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	LX:s	UX:s	LX:s	UX:s
16	2.07754	3.03004	2.0577	5.0
17	2.22657	3.25261	2.17737	3.39486
18	2.41092	3.48039	2.33813	3.65682
19	2.62029	3.75506	2.52718	3.96935
20	2.87882	4.18851	2.76958	4.47515

Table 7: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$,s = 1,...,20, from GLE distribution case I

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	LY:s	UY:s	LY:s	UY:s
1	0.562573	2.00061	0.49741	2.14966
2	1.04009	2.21059	0.958697	2.37733
3	1.32532	2.35793	1.24135	2.53537
4	1.53127	2.47761	1.44764	2.66313
5	1.69538	2.58157	1.6129	2.77385
6	1.83401	2.67558	1.75285	2.87388
7	1.95578	2.763	1.87589	2.96692
8	2.06582	2.84604	1.98706	3.05538
9	2.16746	2.92633	2.08966	3.14106
10	2.26308	3.00521	2.18607	3.22542
11	2.35448	3.08385	2.27807	3.3098
12	2.44313	3.16347	2.36715	3.39554
13	2.53037	3.24538	2.45464	3.48417
14	2.6175	3.33122	2.54184	3.57759
15	2.70597	3.42323	2.63019	3.67843
16	2.79759	3.52475	2.72146	3.79068
17	2.89493	3.64146	2.81818	3.92116
18	3.00218	3.78445	2.92446	4.08332
19	3.12787	3.98114	3.04867	4.31048
20	3.29658	4.33706	3.21517	4.73082

Table 8: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$,s = 1,...,20, from GLE distribution case I for hyper parameters $w_1 = 2, w_2 = 3, w_3 = 1, w_4 = 1, w_5 = 2, w_6 = 3$

MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley$w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	$LY:s$	$UY:s$	$LY:s$	$UY:s$
1	1.24497	3.83974	1.08895	4.04442
2	1.96905	3.97768	1.81567	4.22018
3	2.30715	4.05058	2.15679	4.30927
4	2.51196	4.09936	2.36124	4.36742
5	2.65361	4.13568	2.50105	4.40999
6	2.75956	4.16446	2.60468	4.4433
7	2.843	4.18819	2.68573	4.47049
8	2.91114	4.20832	2.75159	4.49338
9	2.96831	4.22577	2.80668	4.51309
10	3.01728	4.24113	2.85379	4.53036
11	3.05592	4.25484	2.89477	4.54568
12	3.09755	4.26721	2.93094	4.55945
13	3.13112	4.27846	2.96324	4.57193
14	3.16136	4.28877	2.99237	4.58333
15	3.18882	4.29828	3.01885	4.59381
16	3.21391	4.3071	3.04309	4.60351
17	3.23698	4.31532	3.06543	4.61252
18	3.25831	4.32301	3.08612	4.62094
19	3.27811	4.33024	3.10537	4.62882
20	3.29658	4.33705	3.12336	4.63625

Table 9: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$,s = 1,...,20, from GLE distribution case II

MCMC $w_1=w_2=w_3=w_4=w_5=w_6=0$			Lindley $w_1=w_2=w_3=w_4=w_5=w_6=0$	
s	$LY:s$	$UY:s$	$LY:s$	$UY:s$
1	0.562574	2.00062	0.49741	2.14966
2	1.04009	2.2106	0.958697	2.37733
3	1.32532	2.35793	1.24135	2.53537
4	1.53127	2.47761	1.44764	2.66313
5	1.69538	2.58157	1.6129	2.77385
6	1.83401	2.67558	1.75285	2.87388
7	1.95578	2.763	1.87589	2.96692
8	2.06582	2.84604	1.98706	3.05538
9	2.16746	2.92633	2.08966	3.14106
10	2.26308	3.00521	2.18607	3.22542
11	2.35447	3.08385	2.27807	3.3098
12	2.44313	3.16347	2.36715	3.39554
13	2.53037	3.24538	2.45464	3.48417
14	2.6175	3.33123	2.54184	3.57759
15	2.70597	3.42323	2.63019	3.67843
16	2.7976	3.52475	2.72146	3.79068
17	2.89493	3.64146	2.81818	3.92116
18	3.00219	3.78446	2.92446	4.08332
19	3.12787	3.98115	3.04867	4.31048
20	3.29658	4.33705	3.21517	4.73082

Table 10: 95% two-sample Bayesian prediction bounds for $Y_{s,m}$,s = 1,...,20, from GLE distribution case II for hyper parameters $w_1 = 2, w_2 = 3, w_3 = 1, w_4 = 1, w_5 = 2, w_6 = 3$

MCMC $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$			Lindley $w_1=2,w_2=3,w_3=1,w_4=1,w_5=2,w_6=3$	
s	LY:s	UY:s	LY:s	UY:s
1	1.24497	3.83974	1.08895	4.04442
2	1.96905	3.97768	1.81567	4.22018
3	2.30715	4.05058	2.15679	4.30927
4	2.51196	4.09936	2.36124	4.36742
5	2.6536	4.13569	2.50105	4.40999
6	2.75956	4.16446	2.60468	4.4433
7	2.843	4.18819	2.68573	4.47049
8	2.91114	4.20832	2.75159	4.49338
9	2.96831	4.22576	2.80668	4.51309
10	3.01727	4.24113	2.85379	4.53036
11	3.05592	4.25484	2.89477	4.54568
12	3.09755	4.26721	2.93094	4.55945
13	3.13112	4.27846	2.96324	4.57193
14	3.16136	4.28877	2.99237	4.58333
15	3.18882	4.29828	3.01885	4.59381
16	3.21391	4.3071	3.04309	4.60351
17	3.23698	4.31532	3.06543	4.61252
18	3.25831	4.32302	3.08612	4.62094
19	3.27811	4.33022	3.10537	4.62882
20	3.29658	4.33705	3.12336	4.63625

4.1. Conclusion

We assume the real data to have come from the GLE distribution. Based on the above two Type-II hybrid censoring schemes, we construct 95% one-sample Bayesian prediction intervals for future order statistics $X_{s,n}, s = 16,\dots,20,$ from the same sample as well as 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:m}, s = 1,\dots,20,$ from a future sample of size $m = 20$.

From Tables 5-10 the following are observed.

1- From Tales 5-10, it is clear that, the lower bounds are relative insensitive to the specification of the hyper parameters $(w_1, w_2, w_3, w_4, w_5, w_6)$ while the upper bounds are almost insensitive.

2- Moreover, Tables 5-10 show that the Bayesian prediction bounds based on Lindly approximation are more close to each other to the specification of the hyper parameters $(w_1, w_2, w_3, w_4, w_5, w_6)$ than the Bayesian prediction bounds based on MCMC method.

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References

1. AL-Hussaini, E. K. (1999a). Bayesian prediction under a mixture of two exponential components model based on Type-I censoring. *J. of App. Stat. Sci.* 8 173– 185.
2. AL-Hussaini, E. K. (1999b). Predicting observable from a general class of distributions. *J. of Stat. Planning and Inference* 79 79–91.
3. AL-Hussaini, E. K., Nigm, A. M., and Jaheen, Z. F. (2001). Bayesian prediction based on finite mixtures of lomax components model and Type I censoring. *Statistics* 35 259–268.
4. Balakrishnan, N. and Shafay, A. R. (2012). One- and Two -sample Bayesian prediction intervals based on Type-II hybrid censored data. *Commun. Stat. Theory and Methods* 41 1511–1531.
5. Banerjee, A. and Kundu, D. (2008). Inference Based on Type-II Hybrid Censored Data From a Weibull Distribution. *IEEE Trans. on Reliab.* 2 369– 378.
6. Childs, A., Chandrasekar, B., Balakrishnan, N. and Kundu, D. (2003). Exact Likelihood Inference Based on Type-I and Type-II Hybrid Censored Samples from the Exponential Distribution. *Ann. of the Instit. of Stat. Math.* 55 319–330.
7. Draper, N. and Guttman, I. (1987). Bayesian analysis of hybrid life test with exponential

failure times. *Ann. of the Instit. of Stat. Math.* 39 219–225.
8. Dunsmore, I. R. (1974). The Bayesian predictive distribution in life testing models. *Techno metrics* 16 455– 460.
9. Ebrahimi, N. (1992). Prediction intervals for future failures in the exponential distribution under hybrid censoring. *IEEE Transactions Reliability* 41 127–132.
10. Epstein, B. (1954). Truncated life tests in the exponential case. *Ann. of Math. Stat.* 25 555–564.
11. Epstein, B. (1960). Estimation from life-test data. *Techno metrics* 2 447– 454.
12. Gross, A. J. and Clark, V. A. (1975). *Survival distributions: Reliability applications in the biomedical sciences* John Wiley and Sons, New York.
13. Gupta, R. D. and Kundu, D. (1998). Hybrid Censoring Schemes with Exponential Failure Distribution. *Commun Stat. Theory and Methods* 27 3065–3083.
14. Jeong, H. S., Park, J. I. and Yum, B. J. (1996). Development of (r, T) hybrid sampling plans for exponential lifetime distributions. *J. Applied Stat.* 23 601–607.
15. Joarder, A., Krishna, H. and Kundu, D. (2009). On Type-II Progressively Hybrid Censoring. *J. of Math App. Stat. Methods* 8 534 – 546.
16. Kundu, D. (2007). On hybrid censored Weibull distribution. *J. of Stat. Planning and Inference* 137 2127–2142.
17. Kundu, D. and Pradhan, B. (2009). Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring. *Commun. Stat. Theory and Methods* 38 2030–2041.
18. Lindley, D. V. (1980). *Approximate Bayes Methods*, Bayesian Statistics, Valency.
19. Mahmoud, M. A. W. and Alam, F. M. A. (2010). The generalized linear exponential distribution. *Statistics and Probability Letters*. 80 1005–1014.
20. Nigm, A. M. (1989). An informative Bayesian prediction for the Weibull lifetime distribution. *Commun. Stat. Theory and Methods* 18 897–911.
21. Park, S. and Balakrishnan, N. (2009). On simple calculation of the Fisher information in hybrid censoring schemes. *Statistics and Probability Letters* 79 1311–1319.
22. Park, S., Balakrishnan, N. and Zheng, G. (2008). Fisher information in hybrid censored data. *Statistics and Probability Letters* 78 2781–2786.
23. Raqab, M. Z. and Madi, M. T. (2005). Bayesian inference for the generalized exponential distribution. *J. of Statistical Computation and Simulation* 69 109–124.
24. Rastogi, M. K. and Tripathi, Y. M. (2013). Estimation using hybrid censored data from a two-parameter distribution with bathtub shape. *Computational Statistics and Data Analysis* 67 268–281.
25. Shafay, A. R. and Balakrishnan, N. (2012) One- and Two- Sample Bayesian Prediction Intervals Based on Type-I Hybrid Censored Data. *Communications in Statistics - Simulation and Computation* 41 65–88.
26. Singh, S. K., Singh, U. and Sharma, V. K. (2013). Bayesian prediction of future observations from inverse Weibull distribution based on type-II hybrid censored sample. *In. J. of Advanced Statistics and Probability* 1 32– 43.

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