Fuzzy quasi-metric spaces of Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3

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Abstract: Following the ideas of Nagoor Gani and Umamaheswari, a notion of fuzzy detour distance between vertices in FACS is introduced and used to define a fuzzy quasi-metric of FACS of fuzzy graph Type-3 and some of its properties.
[Obaid UQ, Ahmad T. **Fuzzy quasi-metric spaces of Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3.** *Life Sci J* 2014;11(9):738-746]. (ISSN:1097-8135). http://www.lifesciencesite.com. 112

Keywords: Fuzzy graph**,** Fuzzy autocatalytic set**,** Incineration process**,** Fuzzy quasi-metric space.

1. Introduction

In 1965, the fuzzy set theory was introduced by Zadeh. Since that time, many researchers have been concerned with the characteristics and applications of fuzzy sets. Fuzzy graph is an important extension of fuzzy theory's application in its relation to graph theory. Fuzzy graph was introduced by Rosenfeld (Rosenfeld, 1975) and Yeh & Bang (Yeh and Bang, 1975) independently.

An example in the application of fuzzy graph theory is in the modeling of Clinical Waste Incineration Process (Baharum, et al., 2009). The system was initially modelled using crisp graph (Ahmad, et al., 2010). Fuzzy graph provides important tools to take various aspects of complexity, inexactitude and fuzziness of the network structure of the incineration system as compared to the description of relation of its crisp graph. The model was found to be an Autocatalytic Set (ACS), conforming to the key feature of the model proposed by Jain and Krishna (Jain and Krishna, 1998). However, the model is insufficient to explain the process (Ahmad, et al., 2010). Therefore, integration of fuzzy graph into the model has eventually created a new concept known as Fuzzy Autocatalytic Set (FACS) and shown to be a better and improved model in explaining the process (Baharum, et al., 2009) (Ahmad, et al., 2010). Six important variables identified in the process are represented as nodes and the catalytic relationships are represented by fuzzy edges.

In this paper, we study FACS of fuzzy graph Type-3 of an incineration process from a new perspective, namely fuzzy quasi-metric spaces as defined in (Gregori and Romaguera, 2004). The study of FACS from quasi-metric viewpoint can be captured to provide a better interpretation to its structure. In this paper, we initially introduce the notion of fuzzy detour distance between vertices in FACS by following the ideas of Nagoor Gani and Umamaheswari (Nagoor Gani and Umamaheswari, 2011). Then, the concepts of fuzzy detour FT3-eccentricity of a vertex, fuzzy detour FT3-radius, fuzzy detour FT3-diameter, fuzzy detour neighbour of a vertex and fuzzy detour boundary of a vertex in FACS are introduced. We extend these ideas to define fuzzy quasi-metric spaces of FACS of fuzzy graph Type-3 and investigate some of their properties such as a convergent sequence which is used to justify an irreducible graph.

2. Preliminaries

In this section, some basic definitions that are necessary in the paper are reviewed. It began with brief explanations on the development of FACS in particular fuzzy graph of type-3 and its application in modeling clinical waste incineration process, followed by some pertinent fuzzy metric concepts and facts.

A clinical waste incineration process in Malacca (schematic diagram given in Figure 1) was initially modelled using crisp graph (Ahmad, et al., 2010) as in Figure 2. However the interpretation of the graph at the end of the process did not signify the product of the process (Baharum, et al., 2009). Therefore, sharing of fuzzy graph into the model has eventually created a new concept known as Fuzzy Autocatalytic Set (FACS) in particular fuzzy graph of type-3 (Ahmad, et al., 2010). Rosenfeld (Rosenfeld, 1975) has given the notion of fuzzy graph as follows.

Definition 1: A fuzzy graph $G:(\sigma, \mu)$ with a vertex set V as the underlying set is a pair of functions that $\sigma: V \rightarrow [0,1]$ is a fuzzy subset of V and $\mu: V \times V \rightarrow [0,1]$ is a fuzzy relation on V, such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ where $\sigma(u) \wedge \sigma(v)$ denotes the minimum of $\sigma(u)$ and $\sigma(v)$.

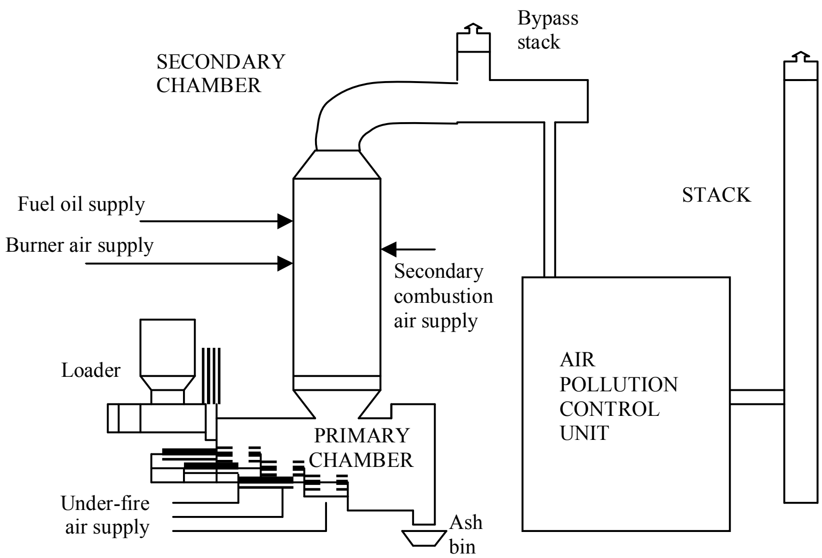

Figure 1. The schematic diagram of a clinical waste incinerator

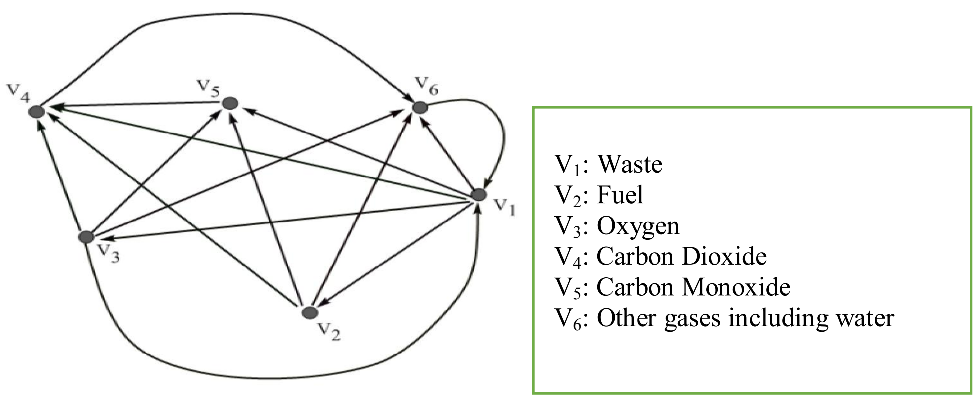

Figure 2. Crisp graph for the clinical waste incineration process

Definition 2: (Rosenfeld, 1975) A path p from a vertex v_i to a vertex v_j in a fuzzy graph is a sequence of distinct vertices and edges starting from v_i and ending at v_j. If v_i and v_j coincide in a path then we call p the cycle.

The underlying crisp graph of the fuzzy graph G:(σ, μ) is denoted as G(V,E) where V is a nonempty set of vertices and E is the nonempty set of edges. Yeh and Bang (Yeh and Bang, 1975) also introduced a special case of graph fuzziness where only the edges are fuzzy and the vertices remain as a crisp set. After fuzzy graphs were introduced by Rosenfeld and Yeh & Bang independently in 1975, Blue et al. (Blue, et al., 1997) (Blue, et al., 2002) further generalized the catalog of various fuzziness possible in graph into five types of fuzzy graphs. Furthermore, Ahmad et al. (Ahmad, et al., 2010)

Formalized the five types of fuzzy graphs described by Blue et al. as follows.

Definition 3: Fuzzy graph is a graph G_F satisfying one of the fuzziness (G_F^i of the ith type) or any of its combination:

1) $G_F^1 = \{ G_{1_F}, G_{2_F}, G_{3_F}, ..., G_{n_F}\}$ where fuzziness is on G_{i_F} for $i=1, 2, 3,..., n$.
2) $G_F^2 = \{V, E_F\}$ where the edge set is fuzzy.
3) $G_F^3 = \{V, E(t_F, h_F)\}$ where both the vertex and edge sets are crisp, but the edges have fuzzy heads and tails.
4) $G_F^4 = \{V_F, E\}$ where the vertex set is fuzzy.
5) $G_F^5 = \{V, E(w_F)\}$ where both the vertex and crisp set are crisp but the edges have fuzzy weights.

The major idea of the notion of FACS is the merger of fuzzy graph of type-3 to autocatalytic set. The definition of FACS is given as follows.

Definition 4: (Ahmad, et al., 2010) Fuzzy autocatalytic set (FACS) is a subgraph where each of whose nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1], \forall\, e_i \in E$.

The membership values for fuzzy edge connectivity for fuzzy graph are in the interval (0, 1]. These values constitute the entries of the adjacency matrix for FACS as follows.

Definition 5: (Baharum, et al., 2009)

Let $C_{F_{ij}}$ denotes fuzzy edge connectivity between node i and node j such that

$$C_{F_{ij}} = \begin{cases} 0 & if\ i = j,\ \ e_i \notin E \\ \mu(e_i\ ) & if\ i \neq j \end{cases}$$

A graph with s nodes is completely specified by an $s\times s$ matrix, $C_{F_{ij}} = (c_{ij})$ called the adjacency matrix of the graph (Harary, 1969). Also a graph C is termed irreducible if each node in the graph has access to every other node (Harary, 1969). According to (Horn and Johnson, 1985) the adjacency matrix $C_{F_{ij}}$ of the graph is irreducible if and only if $(I + C_{F_{ij}})^{s-1} > 0$. Consequently, the adjacency matrix of FACS of fuzzy graph type-3 is irreducible.

Thus, if a graph is irreducible then its associated adjacency matrix is also irreducible and vice versa. Furthermore, the definition of a connected graph (Harary, 1969) which is for every pair of vertices in C are joined by a path. Hence, by Definition of Harary, an irreducible graph is a strongly connected graph and the converse is also true.

As for incineration process, the membership values are determined through the chemical reactions taken place between six variables that play its vital roles in the clinical waste incinerator, namely waste, fuel, oxygen, carbon dioxide, carbon monoxide and other gases including water. The set of vertices in the graph of FACS of the incineration process V={v_1, $v_2,\ldots,v_6$ } is represented by these six variables. From the explanation given in (Baharum, et al., 2009) and (Ahmad, et al., 2010) pertaining to the construction of FACS of Fuzzy Graph of Type-3 for the incineration process, the graph is represented as in Figure 3 and its adjacency matrix given in Definition 5 is represented as in Figure 4.

Next we recall some related fuzzy metric concepts and facts. According to (Romaguera, et al., 2007) a binary operation * : [0, 1]×[0, 1]→ [0,1] is a continuous t-norm if it satisfies the following conditions:

(1) * is associative and commutative,
(2) * is continuous,
(3) a * 1 = a, for all a∈ [0,1],
(4) a * b≤ c*d whenever a ≤c and b≤ d, for each a, b, c, d ∈[0,1].

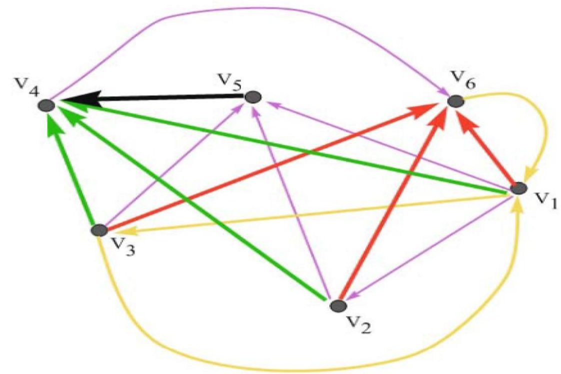

Figure 3. Fuzzy graph of Type-3 for the clinical waste incineration process

$$C_{F_{ij}} = \begin{pmatrix} 0 & 0 & 0.06529 & 0 & 0 & 0.13401 \\ 0.00001 & 0 & 0 & 0 & 0 & 0 \\ 0.15615 & 0 & 0 & 0 & 0 & 0 \\ 0.51632 & 0.68004 & 0.63563 & 0 & 0.99999 & 0 \\ 0.00001 & 0.00001 & 0.00002 & 0 & 0 & 0 \\ 0.32752 & 0.31995 & 0.29906 & 0.00001 & 0 & 0 \end{pmatrix}$$

Figure 4. The adjacency matrix of FACS for the clinical waste incineration process (Baharum, et al., 2009).

Definition 6:(fletcher and Lindgren, 1982)

A quasi-metric on a nonempty set X is a nonnegative real valued function d on X ×X such that for all x, y, z ∈ X:

(i) x = y if and only if d(x, y) = d(y, x) = 0,
(ii) d(x, z) ≤ d(x, y) + d(y, z).

If d satisfies condition (i) above and (ii') d(x, z) ≤ max{d(x, y), d(y, z)} then, d is called a non-Archimedean quasi-metric on X.

If d satisfies the conditions (i), (ii) and (ii'') d(x, y) = d(y, x) then, d is called a metric on X.

The notion of a fuzzy metric space was modified by George and Veeramani in 1994 as follows.

Definition 7: (George and Veeramani, 1994)

A 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary (nonempty) set, * is a continuous t-norm, and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions for each x,y,z ∈ X and t, s > 0:

(1) M (x, y, t) >0,
(2) M (x, y,t) =1 if and only if x=y,
(3) M (x,y,t) = M (y,x,t),
(4)M (x,y,t) * M (y,z,s)≤ M (x,z,t+s),
(5) M (x,y,.): (0,∞)→ (0,1] is continuous.

Remark 8: (George and Veeramani, 1994)

(1)The value M(x, y, t) can be considered as the degree of nearness between x and y with respect to t.

(2) M(x, y, ∗) is nondecreasing for all x, y in X.

The concept of fuzzy quasi-metric space is defined by Gregori and Romaguera in 2004. The researchers generalize the corresponding notion of fuzzy metric space by George and Veeramani (see Definition 7) to the quasi-metric context and is given as follows.

Definition 9: (Gregori and Romaguera, 2004)

A fuzzy quasi-metric space is an ordered triple (X, M, ∗) such that X is a nonempty set, ∗ is a continuous *t*-norm and *M* is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

Q1: M(x, y, t) >0,

Q2: M(x, y, t) = 1 if and only if x = y,

Q3: M(x, y, t) ∗ M(y, z, s) ≤ M(x, z, t +s),

Q4: M(x, y, ·) : (0,∞)→(0, 1] is continuous.

Condition Q2 is equivalent to the following:

M(x, x, t) = 1 for all x ∈ X and t > 0, and M(x, y, t) < 1 for all $x \neq y$ and t > 0.

If (X, M, ∗) is a fuzzy quasi-metric space, we will say that (M, ∗) as a fuzzy quasi-metric on *X*. A fuzzy quasi-metric M is a fuzzy metric, in the sense of George and Veeramani (1994), if M(x, y, t) = M(y, x, t) for all t > 0.

3. The fuzzy detour FT3-distance between vertices in FACS

Following the ideas of Nagoor Gani and Umamaheswari (Nagoor Gani and Umamaheswari, 2011) fuzzy detour distance between vertices in FACS of fuzzy graph Type-3 is defined as follows:

Definition 10: (fuzzy detour FT3-distance between vertices in FACS)

Let $G_{FT3}(V,E)$ be a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3.The fuzzy detour FT3-distance $d_{FT3}(v_i, v_j)$ between two vertices v_i and v_j in FACS is defined as the maximum length of any v_i-v_j path, where the FT3-length of a path p: $v_0,v_1,...,v_n$ is $\ell(p) = \sum_{i=1}^{n} \frac{1}{\mu((v_{i-1},v_i))}$ and $d_{FT3}(v_i, v_j) = 0$ if and only if $v_i = v_j$. A v_i- v_j path of length $d_{FT3}(v_i, v_j)$ is called v_i- v_j fuzzy detour path.

Example 11:

In this example of FACS with the set of vertices V= $\{v_1, v_2, v_3, v_4\}$ and the membership values $\mu(e_i) = \mu((v_i, v_{i+1}))$ for fuzzy edge connectivity of FACS are five values, there are two paths between v_1 and v_4:

(1) $\{\mu((v_1, v_2)), \mu((v_2, v_4))\}$ i.e. $p_1 = \{v_1, v_2, v_4\}$ with length $\ell(p_1) = \frac{1}{0.4} + \frac{1}{0.2} = 7.5$

(2) $\{\mu((v_1, v_2)), \mu((v_2, v_3)), \mu((v_3, v_4))\}$ i.e. $p_2 = \{v_1, v_2, v_3, v_4\}$ with length $\ell(p_2) = \frac{1}{0.4} + \frac{1}{0.3} + \frac{1}{0.1} = 15.83333$

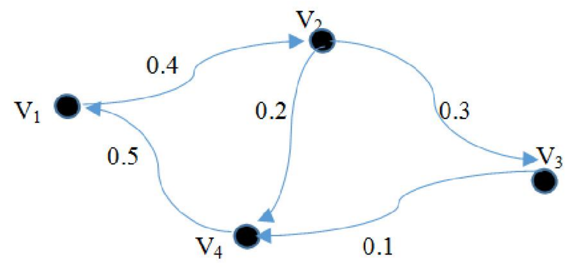

Figure 5. The fuzzy detour FT3-distance $d_{FT3}(v_i, v_j)$ between vertices in FACS

It is clear that $\ell(p_2)$ is the fuzzy detour FT3-distance between v_1 and v_4 and hence $d_{FT3}(v_1, v_4) = 15.83333$ and p_2 is called the v_1- $v_4$fuzzy detour path.

In the following it can be shown that the fuzzy detour FT3-distance is a quasi-metric. To prove it, we first give a lemma to show that the fuzzy detour FT3-distance satisfies the triangle inequality.

Lemma 12: Let $d_{FT3}(v_i, v_j) = a_i > 0$ for all $i \neq j = 1,2,3, \ldots, n$, then

$d_{FT3}(v_1, v_2) + d_{FT3}(v_2, v_3) + \cdots + d_{FT3}(v_{n-1}, v_n) \geq d_{FT3}(v_1, v_n)$.

Proof: (by mathematical induction)

It is clear that $p_1 : d_{FT3}(v_1, v_2) = d_{FT3}(v_1, v_2) = a_1$. Now, assume that

$p_n : d_{FT3}(v_1, v_2) + d_{FT3}(v_2, v_3) + \cdots + d_{FT3}(v_{n-1}, v_n) \geq d_{FT3}(v_1, v_n)$ is true. Then we must show that

$p_{n+1} : d_{FT3}(v_1, v_2) + d_{FT3}(v_2, v_3) + \cdots + d_{FT3}(v_{n-1}, v_n) + d_{FT3}(v_n, v_{n+1}) \geq d_{FT3}(v_1, v_{n+1})$ is true. Note that

$d_{FT3}(v_1, v_2) + d_{FT3}(v_2, v_3) + \cdots + d_{FT3}(v_{n-1}, v_n) + d_{FT3}(v_n, v_{n+1})$

$\geq d_{FT3}(v_1, v_n) + d_{FT3}(v_n, v_{n+1})$ (by p_n)

$\geq d_{FT3}(v_1, v_{n+1})$ (by our assumption). This concludes the proof. □

Theorem 13: Let $G_{FT3}(V,E)$ be a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3. The fuzzy detour FT3-distance $d_{FT3}(v_i, v_j)$ between two vertices v_i and v_j in FACS of fuzzy graph Type-3 is a quasi-metric.

Proof: Since $G_{FT3}(V,E)$ is a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3, therefore the adjacency matrix of FACS is irreducible (Horn and Johnson, 1985) which imply that the graph is strongly connected. Consequently, it is obvious from Definition 10 that $d_{FT3}(v_i, v_j) > 0$ for all v_i, $v_j \in V$ and $d_{FT3}(v_i, v_i) = 0$.

Note that $d_{FT3}(v_i, v_j) \neq d_{FT3}(v_j, v_i)$ due to the graph $G_{FT3}(V,E)$ is a directed graph (or digraph). Now, by Lemma 12, the condition (ii) of Definition 6 is satisfied, i.e.,

$d_{FT3}(v_i, v_j) \leq d_{FT3}(v_i, v_k) + d_{FT3}(v_k, v_j)$ for all v_i, $v_j, v_k \in$ V. Hence the fuzzy detour FT3-distance is a quasi-metric. □

3.1 The fuzzy detour FT3-distance of FACS for the Clinical Waste Incineration Process

In this section, the concepts of fuzzy detour FT3- eccentricity of a vertex, fuzzy detour FT3-radius, fuzzy detour FT3-diameter, fuzzy detour neighbour of a vertex and fuzzy detour boundary of a vertex in FACS are introduced.

Definition 14:(The fuzzy detour FT3- eccentricity of a vertex**)**

Let G_{FT3}(V,E) be a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3.The fuzzy detour FT3-eccentricity e_{FT3}(v) of a vertex v of FACS is the maximum fuzzy detour FT3- distance from any vertex of V to v, i.e. e_{FT3}(v) = max {d_{FT3} (u,v) : u ∈ V(G_{FT3}) – {v}}.

It is obvious that d_{FT3} (u,v) ≤ e_{FT3}(v) for any vertex u of FACS and a vertex u is an eccentric vertex of a vertex v if d_{FT3} (u,v) = e_{FT3}(v).

Definition 15:(The fuzzy detour FT3-radius and diameter of FACS)

The fuzzy detour FT3-radius of FACS is the minimum fuzzy detour FT3- eccentricity among the vertices of FACS. The fuzzy detour FT3- diameter of FACS is the maximum fuzzy detour FT3- eccentricity among the vertices of FACS.

Definition 16:(The fuzzy detour neighbour of a vertex)

Suppose G_{FT3}(V,E) is a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3. For a vertex v in V, define n_{FT3}(v) = min {d_{FT3} (u,v) : u ∈ V(G_{FT3}) – {v}}. A vertex u (≠ v) is called a fuzzy detour neighbour of v if d_{FT3} (u,v)= n_{FT3} (v). The fuzzy detour neighbours of v are denoted by N(v).

As for incineration process (see figure 3), the distance between two vertices in FACS of Clinical Waste Incineration Process can be computed by using the definition of a fuzzy detour FT3-distance on FACS ($d_{FT3}(v_i, v_j)$) and is presented in table 1.

As for FACS of Clinical Waste Incineration Process, we can calculate the following:

e_{FT3}(v_1) = 208.46213, e_{FT3}(v_2) = 258.46213, e_{FT3}(v_3) = 214.86622, e_{FT3} (v_4) =211.80594, e_{FT3} (v_5) = 309.03536, e_{FT3}(v_6) = 301.00001. The description of the clinical incineration process, the fuzzy detour FT3-eccentricity e_{FT3}(v) of a vertex v means that every element (u) has a certain proportion of the interaction with the element v at most with value e_{FT3}(v). Thus, the fuzzy detour FT3-radius of FACS is e_{FT3}(v_1) = 208.46213 and the fuzzy detour FT3- diameter of FACS is e_{FT3}(v_5) = 309.03536.

On the other hand, the fuzzy detour neighbour of vertex when interpreted physically means that every element (u) has a certain proportion of the interaction with the element v at least with value n_{FT3}(v). Hence, we can observe the following:

v_6 is a fuzzy detour neighbour of v_1.
v_1 is a fuzzy detour neighbour of v_2.
v_1 is a fuzzy detour neighbour of v_3.
v_5 is a fuzzy detour neighbour of v_4.
v_1 is a fuzzy detour neighbour of v_5.
v_4 is a fuzzy detour neighbour of v_6.

Definition 17:(The fuzzy detour boundary vertex)

Let G_{FT3}(V,E) be a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3. A vertex v in V is a fuzzy detour boundary vertex of a vertex u if d_{FT3} (u,w) ≤ d_{FT3} (u,v) for every fuzzy detour neighbour w of v. A vertex v is a fuzzy detour boundary vertex of FACS if v is a fuzzy detour boundary of some vertex of FACS.

In FACS of Clinical Waste Incineration Process, we observe that

1. v_6 is a fuzzy detour neighbour of v_1 and v_1 is a fuzzy detour boundary vertex of v_2, v_3, v_4, v_5.
2. v_1 is a fuzzy detour neighbour of v_2 and v_2 is a fuzzy detour boundary vertex of v_3, v_4, v_5, v_6.
3. v_1 is a fuzzy detour neighbour of v_3 and v_3 is a fuzzy detour boundary vertex of v_2, v_4, v_5, v_6.
4. v_5 is a fuzzy detour neighbour of v_4 and v_4 is a fuzzy detour boundary vertex of v_1, v_6 but v_4 is not a fuzzy detour boundary vertex of v_2, v_3.
5. v_1 is a fuzzy detour neighbour of v_5 and v_5 is a fuzzy detour boundary vertex of v_2, v_3, v_4, v_6.
6. v_4 is a fuzzy detour neighbour of v_6 and v_6 is a fuzzy detour boundary vertex of v_1, v_2, v_5 but v_6 is not a fuzzy detour boundary vertex of v_3.

With respect to the fuzzy detour boundary vertex v of a vertex u is that the element u has a certain proportion of the interaction with the element v at most with value e_{FT3}(v) which u is further than neighbour w of v from u in the sense of fuzzy detour distance. This interpretation is presented in the following lemma.

Lemma 18: Let G_{FT3}(V,E) be a no-loop Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3. Every vertex v in V different from u is a fuzzy detour boundary vertex of a vertex u (u ≠ v) if and only if d_{FT3} (u,v) ≥ d_{FT3} (u,w) for every fuzzy detour neighbour w of v.

Proof: Suppose G_{FT3}(V,E) be a graph of FACS of fuzzy graph Type-3, hence the graph is strongly connected (Horn and Johnson, 1985). Then, it is easily seen from Definition 17 that each vertex v in V is different from u (v is a fuzzy detour boundary vertex of a vertex u) ⇔ for every fuzzy detour neighbour w of v, d_{FT3} (u,v) ≥ d_{FT3} (u,w) (u ≠ v).

Table 1. The fuzzy detour FT3-distance between the vertices in FACS of fuzzy graph Type-3 for Clinical Waste Incineration Process

	V_1 (waste)	V_2 (fuel)	V_3 (O_2)	V_4 (CO_2)	V_5 (CO)	V_6 (H_2O&otp*)
V_1 (waste)	0	100	6.40409	201.00001	200	301.00001
V_2 (fuel)	208.46213	0	214.86622	111.58761	208.93262	201.00001
V_3 (O_2)	158.46213	258.46213	0	211.80594	309.03536	151.00001
V_4 (CO_2)	107.46212	207.46212	113.86621	0	307.46212	100
V_5 (CO)	108.46213	208.46213	114.86622	1.00001	0	101.00001
V_6 (H_2O&otp*)	7.46212	107.46212	13.86621	208.46213	207.46212	0

otp* =other pollutants

4. The fuzzy quasi-metric of FACS

In this section, we introduce a fuzzy quasi-metric of FACS of fuzzy graph Type-3. The information about a graph can provide a better insight to its structure by using Definition 21 that will be given in this section. This motivates to the investigation of the structure of FACS using the quasi-metric fuzziness on FACS. The definition will be obtained as a consequence of the following remarks.

Remark 19:(Gregori and Romaguera, 2004) Every quasi-metric on a nonempty set X can induces a fuzzy quasi-metric on X (in the sense of Definition 9). The converse is also true, i.e. every fuzzy quasi-metric generates is a quasi-metrizable topology.

Remark 20:(Gregori and Romaguera, 2004) Let (X, d) be a quasi-metric space. Define a continuous t-norm as $a*b = a.b$ with the usual multiplication for every $a,\ b \in [0,\ 1]$, and let M_d be the function on $X \times X \times (0,\infty)$ defined by $M_d(x, y, t) = \frac{t}{t+d(x,y)}$. Then (X, M_d, $\cdot$) is a fuzzy quasi-metric space and (M_d, $\cdot$) is called the (standard) fuzzy quasi-metric induced by d.

Now, we are in a position to give the notion of a fuzzy quasi-metric of FACS which depends on the fuzzy detour FT3-distance in FACS.

Definition 21: (fuzzy quasi-metric space of FACS) Suppose G_{FT3}(V,E) be a no-loop Fuzzy Autocatalytic Set of fuzzy graph Type-3 and let a continuous t-norm is $a*b = a.b$ the usual multiplication. Let M_{FT3} be a fuzzy function defined by $M_{FT3}(v_i, v_j, t) : V\times V\times (0,\infty) \rightarrow (0,1]$,

$$M_{FT3}(v_i, v_j, t) = \frac{t}{t+d_{FT3}(v_i,v_j)},$$

where t is the number of edges in the v_i- v_j fuzzy detour path p in FACS, $d_{FT3}(v_i, v_j)$ is a fuzzy detour FT3-distance between two vertices v_i and v_j in FACS and M_{FT3} is satisfied the following conditions, for all $v_i, v_j, v_k \in V$ and $t, s > 0$:

1. $M_{FT3}(v_i, v_j, t) > 0$
2. $M_{FT3}(v_i, v_j, t) = 1$ if and only if $v_i = v_j$
3. $M_{FT3}(v_i, v_j, t) * M_{FT3}(v_j, v_k, s) \leq M_{FT3}(v_i, v_k, t+s)$
4. $M_{FT3}(v_i, v_j, .) : (0,\infty) \rightarrow (0, 1]$ is continuous.

Then an ordered triple $(V,\ M_{FT3},\ \cdot)$ is said to be a fuzzy quasi-metric space of FACS and (M_{FT3}, $\cdot$) is called the fuzzy quasi-metric induced by d_{FT3} on FACS.

Table 2. The fuzzy quasi-metric of FACS of fuzzy graph Type-3 for the Clinical Waste Incineration Process

	V_1 (waste)	V_2 (fuel)	V_3 (O_2)	V_4 (CO_2)	V_5 (CO)	V_6 (H_2O&otp*)
V_1 (waste)	1	t = 1 0.00990	t = 1 0.13506	t = 3 0.01470	t = 2 0.00990	t = 4 0.01311
V_2 (fuel)	t = 4 0.01882	1	t = 5 0.02274	t = 4 0.03460	t = 4 0.01878	t = 3 0.01470
V_3 (O_2)	t = 4 0.02462	t = 5 0.01897	1	t = 5 0.02306	t = 5 0.01592	t = 3 0.01948
V_4 (CO_2)	t = 2 0.01827	t = 3 0.01425	t = 3 0.02567	1	t = 4 0.01284	t = 1 0.00990
V_5 (CO)	t = 3 0.02691	t = 4 0.01882	t = 4 0.03365	t = 1 0.49999	1	t = 2 0.01941
V_6 (H_2O&otp*)	t = 1 0.11817	t = 2 0.01827	t = 2 0.12605	t = 4 0.01882	t = 3 0.01425	1

otp* =other pollutants

It is denoted $M_{FT3}(v_i, v_j, t)$ as $M_{FT3}(v_i, v_j)$ and we will say that M_{FT3} is a fuzzy quasi-metric, or simply that (V, M_{FT3}) is a fuzzy quasi-metric space of FACS. Note that M_{FT3} is satisfied the four conditions in Definition 21 by Remarks 19 and 20.

Next, we investigate some of the properties of the fuzzy quasi-metric space of FACS such as a convergent sequence which is used to define an irreducible graph.

Definition 22: (convergent sequence in a fuzzy quasi-metric space (V, M_{FT3}) of FACS)

A sequence $\{v_k\}$ in a fuzzy quasi-metric space (V, M_{FT3}) of FACS is converges to v_0 ($i.e. v_k \to v_o$) if there exists fuzzy detour path ρ: v_k,v_{k-1},...,v_0 of the maximum length d_{FT3} (v_k , v_0) such that $\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$ for all $t > 0$.

Lemma 23: Let $G_{FT3}(V,E)$ be a graph of FACS of fuzzy graph Type-3 and let $\{v_k\}$ be a sequence in a fuzzy quasi-metric space (V, M_{FT3}) of FACS which converges to v_0. Then

$\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$ for all $t > 0$ if and only if $\lim_{k\to\infty} \{d_{FT3}\,(v_k, v_0): v_k \in V(G_{FT3}) - \{v_0\}\} = 0$.

Proof: Suppose $\{v_k\}$ is a sequence in a fuzzy quasi-metric space (V, M_{FT3}) of FACS converges to v_0. Hence by definition 22, there exists fuzzy detour path ρ: v_k,v_{k-1},...,v_0 of the maximum length $d_{FT3}(v_k, v_0)$ and $\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$ for all $t > 0$. This imply that

$\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$ for all $t > 0$ ……(1)

$\Leftrightarrow \lim_{k\to\infty} \frac{t}{t+d_{FT3}(v_k,v_0)} = 1$ (By definition 21)

$\Leftrightarrow \frac{t}{t+\lim_{k\to\infty} d_{FT3}(v_k,v_0)} = 1$

$\Leftrightarrow \frac{t}{t+0} = 1$ for all $t > 0$.

Therefore, $\lim_{k\to\infty} \{d_{FT3}\,(v_k, v_0): v_k \in V(G_{FT3}) - \{v_0\}\} = 0$ and this concludes the proof. □

In other words, by Definition 17, it is shown that $\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1 \Leftrightarrow v_m$ is the last value satisfied equation (1) which is a fuzzy detour neighbor of v_0 and v_0 is a fuzzy detour boundary vertex of $v_k, k \neq m, k \in \Lambda$.

Theorem 24: Let the ordered pair (V, M_{FT3}) be a fuzzy quasi-metric space of FACS. Any graph of FACS of fuzzy graph Type-3 is strongly connected (irreducible graph) if and only if $\forall\, v_o \in V, \exists$ a sequence $\{v_k\}$ in a fuzzy quasi-metric space (V, M_{FT3}) of FACS converges to v_o ($i.e. v_k \to v_o$).

Proof: Let $G_{FT3}(V,E)$ be a graph of FACS of fuzzy graph Type-3 and is strongly connected. Therefore, it is clear that for every vertex v_o, there exists fuzzy detour path between v_o and each other vertex (say $v_k, k \in \Lambda$) ρ: v_k,v_{k-1},...,v_0 with maximum length d_{FT3} (v_k , v_0). Now, it is left to show that $\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$. However, this is satisfied by Lemma 23. Hence, by Definition 22, ∃ a sequence $\{v_k\}$ in a fuzzy quasi-metric space (V, M_{FT3}) of FACS converges to $v_o, \forall\, v_o \in V$.

Conversely, suppose for every $v_o \in V, \exists$ a sequence $\{v_k\}$ in a fuzzy quasi-metric space of FACS converges to v_o ($i.e. v_k \to v_o$) such that $\lim_{k\to\infty} M_{FT3}\,(v_k, v_0, t) = 1$ for all $t > 0$. Hence, there exists fuzzy detour path between any vertex v_o and another vertex v_n in $G_{FT3}(V,E)$ of the graph of FACS. Therefore, the graph of FACS of fuzzy graph Type-3 is clearly strongly connected. □

4.1 The fuzzy quasi-metric of FACS for the Clinical Waste Incineration Process

Let M_{FT3} be a fuzzy function on V×V× (0,∞) defined by $M_{FT3}\,(v_i, v_j, t) = \frac{t}{t+d_{FT3}(v_i,v_j)}$ where t is the number of edges in the v_i- v_j fuzzy detour path p in FACS for the clinical incineration process and $M_{FT3}\,(v_i, v_j)$ is shown in the table 2.

It is clearly seen that the above values of $M_{FT3}\,(v_i, v_j, t)$ is fulfilled (see definition 21). This means that $0 < M_{FT3}\,(v_i, v_j, t) \leq 1, \forall\, v_i, v_j \in V$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $M_{FT3}\,(v_i, v_i, t) = 1$. It is easily verified that $M_{FT3}\,(v_i, v_j, .)$ is a continuous function and M_{FT3} satisfy the triangle inequality i.e.

$M_{FT3}\,(v_i, v_j, t). M_{FT3}\,(v_j, v_k, s) \leq M_{FT3}\,(v_i, v_k, t+s)$ for all $v_i, v_j, v_k \in V$, namely,

$M_{FT3}\,(v_1, v_3, 1). M_{FT3}\,(v_3, v_2, 5) \leq M_{FT3}\,(v_1, v_2, 1)$
$M_{FT3}\,(v_1, v_4, 3). M_{FT3}\,(v_4, v_2, 3) \leq M_{FT3}\,(v_1, v_2, 1)$
$M_{FT3}\,(v_1, v_5, 2). M_{FT3}\,(v_5, v_2, 4) \leq M_{FT3}\,(v_1, v_2, 1)$
$M_{FT3}\,(v_1, v_6, 4). M_{FT3}\,(v_6, v_2, 2) \leq M_{FT3}\,(v_1, v_2, 1)$

$M_{FT3}\,(v_1, v_2, 1). M_{FT3}\,(v_2, v_3, 5) \leq M_{FT3}\,(v_1, v_3, 1)$
$M_{FT3}\,(v_1, v_4, 3). M_{FT3}\,(v_4, v_3, 3) \leq M_{FT3}\,(v_1, v_3, 1)$
$M_{FT3}\,(v_1, v_5, 2). M_{FT3}\,(v_5, v_3, 4) \leq M_{FT3}\,(v_1, v_3, 1)$
$M_{FT3}\,(v_1, v_6, 4). M_{FT3}\,(v_6, v_3, 2) \leq M_{FT3}\,(v_1, v_3, 1)$

$M_{FT3}\,(v_1, v_2, 1). M_{FT3}\,(v_2, v_4, 4) \leq M_{FT3}\,(v_1, v_4, 3)$
$M_{FT3}\,(v_1, v_3, 1). M_{FT3}\,(v_3, v_4, 5) \leq M_{FT3}\,(v_1, v_4, 3)$
$M_{FT3}\,(v_1, v_5, 2). M_{FT3}\,(v_5, v_4, 1) \leq M_{FT3}\,(v_1, v_4, 3)$
$M_{FT3}\,(v_1, v_6, 4). M_{FT3}\,(v_6, v_4, 4) \leq M_{FT3}\,(v_1, v_4, 3)$

$M_{FT3}\,(v_1, v_2, 1). M_{FT3}\,(v_2, v_5, 4) \leq M_{FT3}\,(v_1, v_5, 2)$
$M_{FT3}\,(v_1, v_3, 1). M_{FT3}\,(v_3, v_5, 5) \leq M_{FT3}\,(v_1, v_5, 2)$
$M_{FT3}\,(v_1, v_4, 3). M_{FT3}\,(v_4, v_5, 4) \leq M_{FT3}\,(v_1, v_5, 2)$
$M_{FT3}\,(v_1, v_6, 4). M_{FT3}\,(v_6, v_5, 3) \leq M_{FT3}\,(v_1, v_5, 2)$

$M_{FT3}\,(v_1, v_2, 1). M_{FT3}\,(v_2, v_6, 3) \leq M_{FT3}\,(v_1, v_6, 4)$
$M_{FT3}\,(v_1, v_3, 1). M_{FT3}\,(v_3, v_6, 3) \leq M_{FT3}\,(v_1, v_6, 4)$
$M_{FT3}\,(v_1, v_4, 3). M_{FT3}\,(v_4, v_6, 1) \leq M_{FT3}\,(v_1, v_6, 4)$
$M_{FT3}\,(v_1, v_5, 2). M_{FT3}\,(v_5, v_6, 2) \leq M_{FT3}\,(v_1, v_6, 4)$

$M_{FT3}\,(v_2, v_3, 5). M_{FT3}\,(v_3, v_1, 4) \leq M_{FT3}\,(v_2, v_1, 4)$
$M_{FT3}\,(v_2, v_4, 4). M_{FT3}\,(v_4, v_1, 2) \leq M_{FT3}\,(v_2, v_1, 4)$
$M_{FT3}\,(v_2, v_5, 4). M_{FT3}\,(v_5, v_1, 3) \leq M_{FT3}\,(v_2, v_1, 4)$
$M_{FT3}\,(v_2, v_6, 3). M_{FT3}\,(v_6, v_1, 1) \leq M_{FT3}\,(v_2, v_1, 4)$

$M_{FT3}(v_2, v_1, 4). M_{FT3}(v_1, v_3, 1) \leq M_{FT3}(v_2, v_3, 5)$
$M_{FT3}(v_2, v_4, 4). M_{FT3}(v_4, v_3, 3) \leq M_{FT3}(v_2, v_3, 5)$
$M_{FT3}(v_2, v_5, 4). M_{FT3}(v_5, v_3, 4) \leq M_{FT3}(v_2, v_3, 5)$
$M_{FT3}(v_2, v_6, 3). M_{FT3}(v_6, v_3, 2) \leq M_{FT3}(v_2, v_3, 5)$

$M_{FT3}(v_2, v_1, 4). M_{FT3}(v_1, v_4, 3) \leq M_{FT3}(v_2, v_4, 4)$
$M_{FT3}(v_2, v_3, 5). M_{FT3}(v_3, v_4, 5) \leq M_{FT3}(v_2, v_4, 4)$
$M_{FT3}(v_2, v_5, 4). M_{FT3}(v_5, v_4, 1) \leq M_{FT3}(v_2, v_4, 4)$
$M_{FT3}(v_2, v_6, 3). M_{FT3}(v_6, v_4, 4) \leq M_{FT3}(v_2, v_4, 4)$

$M_{FT3}(v_2, v_1, 4). M_{FT3}(v_1, v_5, 2) \leq M_{FT3}(v_2, v_5, 4)$
$M_{FT3}(v_2, v_3, 5). M_{FT3}(v_3, v_5, 5) \leq M_{FT3}(v_2, v_5, 4)$
$M_{FT3}(v_2, v_4, 4). M_{FT3}(v_4, v_5, 4) \leq M_{FT3}(v_2, v_5, 4)$
$M_{FT3}(v_2, v_6, 3). M_{FT3}(v_6, v_5, 3) \leq M_{FT3}(v_2, v_5, 4)$

$M_{FT3}(v_2, v_1, 4). M_{FT3}(v_1, v_6, 4) \leq M_{FT3}(v_2, v_6, 3)$
$M_{FT3}(v_2, v_3, 5). M_{FT3}(v_3, v_6, 3) \leq M_{FT3}(v_2, v_6, 3)$
$M_{FT3}(v_2, v_4, 4). M_{FT3}(v_4, v_6, 1) \leq M_{FT3}(v_2, v_6, 3)$
$M_{FT3}(v_2, v_5, 4). M_{FT3}(v_5, v_6, 2) \leq M_{FT3}(v_2, v_6, 3)$

$M_{FT3}(v_3, v_2, 5). M_{FT3}(v_2, v_1, 4) \leq M_{FT3}(v_3, v_1, 4)$
$M_{FT3}(v_3, v_4, 5). M_{FT3}(v_4, v_1, 2) \leq M_{FT3}(v_3, v_1, 4)$
$M_{FT3}(v_3, v_5, 5). M_{FT3}(v_5, v_1, 3) \leq M_{FT3}(v_3, v_1, 4)$
$M_{FT3}(v_3, v_6, 3). M_{FT3}(v_6, v_1, 1) \leq M_{FT3}(v_3, v_1, 4)$

$M_{FT3}(v_3, v_1, 4). M_{FT3}(v_1, v_2, 1) \leq M_{FT3}(v_3, v_2, 5)$
$M_{FT3}(v_3, v_4, 5). M_{FT3}(v_4, v_2, 3) \leq M_{FT3}(v_3, v_2, 5)$
$M_{FT3}(v_3, v_5, 5). M_{FT3}(v_5, v_2, 4) \leq M_{FT3}(v_3, v_2, 5)$
$M_{FT3}(v_3, v_6, 3). M_{FT3}(v_6, v_2, 2) \leq M_{FT3}(v_3, v_2, 5)$

$M_{FT3}(v_3, v_1, 4). M_{FT3}(v_1, v_4, 3) \leq M_{FT3}(v_3, v_4, 5)$
$M_{FT3}(v_3, v_2, 5). M_{FT3}(v_2, v_4, 4) \leq M_{FT3}(v_3, v_4, 5)$
$M_{FT3}(v_3, v_5, 5). M_{FT3}(v_5, v_4, 1) \leq M_{FT3}(v_3, v_4, 5)$
$M_{FT3}(v_3, v_6, 3). M_{FT3}(v_6, v_4, 4) \leq M_{FT3}(v_3, v_4, 5)$

$M_{FT3}(v_3, v_1, 4). M_{FT3}(v_1, v_5, 2) \leq M_{FT3}(v_3, v_5, 5)$
$M_{FT3}(v_3, v_2, 5). M_{FT3}(v_2, v_5, 4) \leq M_{FT3}(v_3, v_5, 5)$
$M_{FT3}(v_3, v_4, 5). M_{FT3}(v_4, v_5, 4) \leq M_{FT3}(v_3, v_5, 5)$
$M_{FT3}(v_3, v_6, 3). M_{FT3}(v_6, v_5, 3) \leq M_{FT3}(v_3, v_5, 5)$

$M_{FT3}(v_3, v_1, 4). M_{FT3}(v_1, v_6, 4) \leq M_{FT3}(v_3, v_6, 3)$
$M_{FT3}(v_3, v_2, 5). M_{FT3}(v_2, v_6, 3) \leq M_{FT3}(v_3, v_6, 3)$
$M_{FT3}(v_3, v_4, 5). M_{FT3}(v_4, v_6, 1) \leq M_{FT3}(v_3, v_6, 3)$
$M_{FT3}(v_3, v_5, 5). M_{FT3}(v_5, v_6, 2) \leq M_{FT3}(v_3, v_6, 3)$

$M_{FT3}(v_4, v_2, 3). M_{FT3}(v_2, v_1, 4) \leq M_{FT3}(v_4, v_1, 2)$
$M_{FT3}(v_4, v_3, 3). M_{FT3}(v_3, v_1, 4) \leq M_{FT3}(v_4, v_1, 2)$
$M_{FT3}(v_4, v_5, 4). M_{FT3}(v_5, v_1, 3) \leq M_{FT3}(v_4, v_1, 2)$
$M_{FT3}(v_4, v_6, 1). M_{FT3}(v_6, v_1, 1) \leq M_{FT3}(v_4, v_1, 2)$

$M_{FT3}(v_4, v_1, 2). M_{FT3}(v_1, v_2, 1) \leq M_{FT3}(v_4, v_2, 3)$
$M_{FT3}(v_4, v_3, 3). M_{FT3}(v_3, v_2, 5) \leq M_{FT3}(v_4, v_2, 3)$
$M_{FT3}(v_4, v_5, 4). M_{FT3}(v_5, v_2, 4) \leq M_{FT3}(v_4, v_2, 3)$
$M_{FT3}(v_4, v_6, 1). M_{FT3}(v_6, v_2, 2) \leq M_{FT3}(v_4, v_2, 3)$

$M_{FT3}(v_4, v_1, 2). M_{FT3}(v_1, v_3, 1) \leq M_{FT3}(v_4, v_3, 3)$
$M_{FT3}(v_4, v_2, 3). M_{FT3}(v_2, v_3, 5) \leq M_{FT3}(v_4, v_3, 3)$
$M_{FT3}(v_4, v_5, 4). M_{FT3}(v_5, v_3, 4) \leq M_{FT3}(v_4, v_3, 3)$
$M_{FT3}(v_4, v_6, 1). M_{FT3}(v_6, v_3, 2) \leq M_{FT3}(v_4, v_3, 3)$

$M_{FT3}(v_4, v_1, 2). M_{FT3}(v_1, v_5, 2) \leq M_{FT3}(v_4, v_5, 4)$
$M_{FT3}(v_4, v_2, 3). M_{FT3}(v_2, v_5, 4) \leq M_{FT3}(v_4, v_5, 4)$
$M_{FT3}(v_4, v_3, 3). M_{FT3}(v_3, v_5, 5) \leq M_{FT3}(v_4, v_5, 4)$
$M_{FT3}(v_4, v_6, 1). M_{FT3}(v_6, v_5, 3) \leq M_{FT3}(v_4, v_5, 4)$

$M_{FT3}(v_4, v_1, 2). M_{FT3}(v_1, v_6, 4) \leq M_{FT3}(v_4, v_6, 1)$
$M_{FT3}(v_4, v_2, 3). M_{FT3}(v_2, v_6, 3) \leq M_{FT3}(v_4, v_6, 1)$
$M_{FT3}(v_4, v_3, 3). M_{FT3}(v_3, v_6, 3) \leq M_{FT3}(v_4, v_6, 1)$
$M_{FT3}(v_4, v_5, 4). M_{FT3}(v_5, v_6, 2) \leq M_{FT3}(v_4, v_6, 1)$

$M_{FT3}(v_5, v_2, 4). M_{FT3}(v_2, v_1, 4) \leq M_{FT3}(v_5, v_1, 3)$
$M_{FT3}(v_5, v_3, 4). M_{FT3}(v_3, v_1, 4) \leq M_{FT3}(v_5, v_1, 3)$
$M_{FT3}(v_5, v_4, 1). M_{FT3}(v_4, v_1, 2) \leq M_{FT3}(v_5, v_1, 3)$
$M_{FT3}(v_5, v_6, 2). M_{FT3}(v_6, v_1, 1) \leq M_{FT3}(v_5, v_1, 3)$

$M_{FT3}(v_5, v_1, 3). M_{FT3}(v_1, v_2, 1) \leq M_{FT3}(v_5, v_2, 4)$
$M_{FT3}(v_5, v_3, 4). M_{FT3}(v_3, v_2, 5) \leq M_{FT3}(v_5, v_2, 4)$
$M_{FT3}(v_5, v_4, 1). M_{FT3}(v_4, v_2, 3) \leq M_{FT3}(v_5, v_2, 4)$
$M_{FT3}(v_5, v_6, 2). M_{FT3}(v_6, v_2, 2) \leq M_{FT3}(v_5, v_2, 4)$

$M_{FT3}(v_5, v_1, 3). M_{FT3}(v_1, v_3, 1) \leq M_{FT3}(v_5, v_3, 4)$
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$M_{FT3}(v_5, v_4, 1). M_{FT3}(v_4, v_3, 3) \leq M_{FT3}(v_5, v_3, 4)$
$M_{FT3}(v_5, v_6, 2). M_{FT3}(v_6, v_3, 2) \leq M_{FT3}(v_5, v_3, 4)$

$M_{FT3}(v_5, v_1, 3). M_{FT3}(v_1, v_4, 3) \leq M_{FT3}(v_5, v_4, 1)$
$M_{FT3}(v_5, v_2, 4). M_{FT3}(v_2, v_4, 4) \leq M_{FT3}(v_5, v_4, 1)$
$M_{FT3}(v_5, v_3, 4). M_{FT3}(v_3, v_4, 5) \leq M_{FT3}(v_5, v_4, 1)$
$M_{FT3}(v_5, v_6, 2). M_{FT3}(v_6, v_4, 4) \leq M_{FT3}(v_5, v_4, 1)$

$M_{FT3}(v_5, v_1, 3). M_{FT3}(v_1, v_6, 4) \leq M_{FT3}(v_5, v_6, 2)$
$M_{FT3}(v_5, v_2, 4). M_{FT3}(v_2, v_6, 3) \leq M_{FT3}(v_5, v_6, 2)$
$M_{FT3}(v_5, v_3, 4). M_{FT3}(v_3, v_6, 3) \leq M_{FT3}(v_5, v_6, 2)$
$M_{FT3}(v_5, v_4, 1). M_{FT3}(v_4, v_6, 1) \leq M_{FT3}(v_5, v_6, 2)$

$M_{FT3}(v_6, v_2, 2). M_{FT3}(v_2, v_1, 4) \leq M_{FT3}(v_6, v_1, 1)$
$M_{FT3}(v_6, v_3, 2). M_{FT3}(v_3, v_1, 4) \leq M_{FT3}(v_6, v_1, 1)$
$M_{FT3}(v_6, v_4, 4). M_{FT3}(v_4, v_1, 2) \leq M_{FT3}(v_6, v_1, 1)$
$M_{FT3}(v_6, v_5, 3). M_{FT3}(v_5, v_1, 3) \leq M_{FT3}(v_6, v_1, 1)$

$M_{FT3}(v_6, v_1, 1). M_{FT3}(v_1, v_2, 1) \leq M_{FT3}(v_6, v_2, 2)$
$M_{FT3}(v_6, v_3, 2). M_{FT3}(v_3, v_2, 5) \leq M_{FT3}(v_6, v_2, 2)$
$M_{FT3}(v_6, v_4, 4). M_{FT3}(v_4, v_2, 3) \leq M_{FT3}(v_6, v_2, 2)$
$M_{FT3}(v_6, v_5, 3). M_{FT3}(v_5, v_2, 4) \leq M_{FT3}(v_6, v_2, 2)$

$M_{FT3}(v_6, v_1, 1). M_{FT3}(v_1, v_3, 1) \leq M_{FT3}(v_6, v_3, 2)$
$M_{FT3}(v_6, v_2, 2). M_{FT3}(v_2, v_3, 5) \leq M_{FT3}(v_6, v_3, 2)$
$M_{FT3}(v_6, v_4, 4). M_{FT3}(v_4, v_3, 3) \leq M_{FT3}(v_6, v_3, 2)$
$M_{FT3}(v_6, v_5, 3). M_{FT3}(v_5, v_3, 4) \leq M_{FT3}(v_6, v_3, 2)$

$M_{FT3}(v_6, v_1, 1). M_{FT3}(v_1, v_4, 3) \leq M_{FT3}(v_6, v_4, 4)$
$M_{FT3}(v_6, v_2, 2). M_{FT3}(v_2, v_4, 4) \leq M_{FT3}(v_6, v_4, 4)$
$M_{FT3}(v_6, v_3, 2). M_{FT3}(v_3, v_4, 5) \leq M_{FT3}(v_6, v_4, 4)$
$M_{FT3}(v_6, v_5, 3). M_{FT3}(v_5, v_4, 1) \leq M_{FT3}(v_6, v_4, 4)$

$M_{FT3}(v_6, v_1, 1). M_{FT3}(v_1, v_5, 2) \leq M_{FT3}(v_6, v_5, 3)$
$M_{FT3}(v_6, v_2, 2). M_{FT3}(v_2, v_5, 4) \leq M_{FT3}(v_6, v_5, 3)$
$M_{FT3}(v_6, v_3, 2). M_{FT3}(v_3, v_5, 5) \leq M_{FT3}(v_6, v_5, 3)$
$M_{FT3}(v_6, v_4, 4). M_{FT3}(v_4, v_5, 4) \leq M_{FT3}(v_6, v_5, 3)$

5. Conclusion

This paper explored the realm of FACS of fuzzy graph Type-3 in its relation to the quasi-metric fuzziness. A new concept namely fuzzy quasi-metric space of FACS of fuzzy graph Type-3 is defined and implemented in the modeling of the incineration process of Ahmad et al. We presented the notion of convergent sequence in fuzzy quasi-metric space of FACS and used it to justify an irreducibility of its graph.

Acknowledgements

This research is supported by Research University Grant (GUP 04H94) awarded by Universiti Teknologi Malaysia (UTM). The researchers are thankful to its financial support.

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8/28/2014