Comparison of Two Main Parametric Methods in Multi-Portfolio Optimization

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Abstract: There are many applications for Multi-portfolio optimization in finance, management, engineering and etc. The main problem in this area is to find out the optimal method to distribute a given funds on a set of existing assets. Two methods of multi-portfolio optimization methods were proposed recently: weighted sum method and ε-constraint method. The former is based on weighting, by positive coefficients. The second method considers the one of the objective functions and let others be the constraints. It is found that unlike the weighting method, the scaling of the objective functions is not necessary in the e-constrained method. Finally, some results and discussion are provided in the concluding section. The results compare two main parametric methods of multi-portfolio optimization.

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1. Introduction

In finance, a portfolio is a set of assets that can make one compound from several positions in investing. The main problem in this area is to find out the optimal method to distribute a given fund on a set of existing assets. Maximization of expected return and minimization of risk are two main aims of this problem. The user's risk aversion has a direct effect on the optimal solution. Two criteria are necessary for optimization of the portfolio. The first criterion is the set of solution to the portfolio optimization problem called ''efficient frontier" or ''Pareto-optimal front". The second one is the measurement against the risk of the portfolio. It is a norm that a financial institution wanted to present its customers different position to choose in relation to their risk aversion (Mavrotas, 2009) (Lozza et al., 2011).

The outline of this paper is as follows. In Section 2, it presents some primary concepts that are needed. In Section 3, the weighted sum method as one of the parametric portfolio optimization based on literature was studied. Next, the ε-constraint method was discussed. Section 5 concludes the paper with the comparison and discussion.

2. Primaries of Portfolio Optimization

There are many applications for Multi-objective Optimization (MO). However, some time it is very hard to find the solution of the MO. On the other hand, finding out the all Pareto optimal solution is costly and consumes much time. Furthermore, in some cases the MO problem has an unlimited set of Pareto (Chinchuluun and Pardalos, 2007) (Elahi and Abd Aziz, 2011).

From the theoretical aspect, every Pareto optimal solution is equally satisfying as the solution to the MO problem. However, from a practical point of view, only one reasonable solution must be chosen finally. A decision maker (DM) is a person who can select a good point out of the set of Pareto optimal solutions. A DM has understanding about a set of Pareto optimal solution and can find it from several solutions (Yang et al., 2011, Grodzevich and Romanko, 2006). Generally a MO problem is as the following mathematical model

$$\min f(x) = (f_1(x), \dots, f_p(x)) \quad (1)$$
$$\text{subject to } g(x) \leq 0,$$

where $: \mathbb{R}^n \to \mathbb{R}^p$, $g(x) : \mathbb{R}^n \to \mathbb{R}^m$. Note that f is a vector-valued objective function. For example in the integer programming cases, it is assumed that f and g are linear functions and MO problem with integer program is as the following

$$\min f(x) = C_x \quad (2)$$
$$\text{Subject to } Ax \leq b$$
$$x \in \{0,1\}^n.$$

The feasible set in decision problem can be represented by

$$X = \{x \in \mathbb{R}^n : g(x) \leq 0\}$$
or $X = \{x \in \{0,1\}^n : Ax \leq b\}$

and the feasible set for objective problem can be denoted by

$$Y = f(X) = \{ f(x) : x \in X\}.$$

A feasible solution $\hat{x}$ is called efficient if it satisfies two following conditions:

There is no $x \in X$ with $f(x) \leq f(\hat{x})$ and $f(x) \neq f(\hat{x})$.

So, the image of the efficient set in MO space is equal to one set that is non-dominated point $Y_N := f(X_E)$ (Ehrgott, 2009) (Elahi and Mohd-Ismail, 2012) (Wang and Wu, 2011).

The purpose of Pareto optimality is to find out the best set of solution for MO Problem. This objective vector is such that none of the components of each of those vectors can be better without deterioration to at least one of the other components of the vector. Thus, the mathematical view of the Pareto optimality can be as follows:

Portfolio optimization was introduced by Markowitz (1952) via a framework of return / variance risk (Yu, 2012).

Nowadays, most problems of optimization use multi-objective model (Eichfelder, 2009). Usually the problem of portfolio optimization includes n securities, a preliminary sum to be invested, an opening of a holding time and an end of the holding period (Steuer et al., 2006).

3. Weighted Sum Method

This method was introduced by Zadeh and it is one of the main ways of solving the Multi objectives (Chinchuluun and Pardalos, 2007) (Zadeh and Desoer, 1963). There are three types of this method used for multi-portfolio optimization (MPO) in previous research. They are weighted sum method, weight quadratic method and weight quadratic variation.

One of the main ways of weighting methods is the weighted sum method. The aim of weighting method is the optimization of the objective functions that they are arranged by linear combination (weighted sum). Different efficient solutions can be found by changing the weights of the objective functions (Kirytopoulos et al., 2010). However, Insert the values of weighted coefficients can be wrapped. In fact, the difficulty is to determinate the weighted in regards to each objective function. There is a relation between weights and its corresponding objective function (Grodzevich and Romanko, 2006).

The weighted sum method changes the MO problem with a single model of mathematical optimization problem. In this method, sum weighting coefficient w_i multiplied each objective function f_i to make the structure of the objective function as the following: (note that normalization for those coefficients is not necessary)

$$\min \sum_{i=1}^{k} \omega_i f_i(x) \quad (3)$$
$$\text{s.t. } x \in \Omega,$$
$$\omega_i \geq 0, \forall i = 1, \dots, k$$
$$\sum_{i=1}^{k} \omega_i = 1.$$

With convexity supposition, if $\omega_i > 0$, $\forall i = 1, \dots, k$ then the solution of the above system is Pareto optimal. This means that if the system is convex, then any Pareto optimal solution can be found (Grodzevich and Romanko, 2006). There are three criteria to measure the weight. They are subjective preference of the decision-makers, the variance measure and the independence of criteria. Usually two methods can be used to achieve this aim: the equal weights and the rank-order weights (Wang et al., 2009, Jia, 1997) (Sawik, 2009).

Furthermore, adaptive weighted sum (AWS) method by Kim and Weck is presented in this section. This method considers m-dimensional problems with some constraints which allow us to find the Pareto optimal set.

They extended the bi-objective method to AWS method for solving the problems with which they are multi-objective functions. To achieve this aim, the current weighted sum method is used to approximate the Pareto optimal set, and identified the situation of the Pareto front patches which they are used to refined imposing additional equality constraints to find the m-dimensional objective space (Kim and Viens, 2012).

4. ε-constraint Method

Let MO problem be as the following:

$$\max \left(f_1(x), \dots, f_p(x)\right) \quad (4)$$
$$\text{S.t. } x \in S,$$

Where the vector of decision variables is shown by x, and $f_1(x), \dots, f_p(x)$ are objective functions and the feasible region is denoted by S. In this method, we consider one of the objective functions to the optimization and constraints of this system are other objective functions as the following

$$\max f_1(x), \quad (5)$$
$$f_2(x) \geq e_2,$$
$$f_3(x) \geq e_3,$$
$$\dots$$
$$f_p(x) \geq e_p,$$
$$\text{St } x \in S,$$

The efficient solutions of the problem above come from parametric variation method (Cohon, 1978, Mavrotas, 2009) (Kirytopoulos et al., 2010, Cohon, 1978, Miettinen, 1999). Generation of the complete (continuous) efficient frontier is related to convexity of the portfolio selection. If the portfolio problem is convex, then the solution can be obtained by the critical line algorithm. On the other hand, in non-convex cases, usually that is based on the ε-constraint method.

Obviously, this method helps to generate the single solution and there are some algorithms (e.g. Evolutionary algorithms) that can be simulated for multiple objectives to generate the approximation of the efficient frontier (Mavrotas, 2009) (Miettinen, 1999). The families of ε-constraint formulations are considered as the multiple criteria optimization problem to generate this method.

According to the constraints of objective functions, the ranges of them at least of the $p-1$ objective functions can be determined in the ε-constraint method. Next, with the slack variable the objective function constraints change to equalities. On the other hand, the sum of these slack variables uses to produce effective solutions. It is used to find the possible situation of max $f_1(x)$ based on the one that maximizes the sum as the following:

$$\max(f_1(x) + \delta \times (s_2 + s_3 + \cdots + s_p))$$
$$\text{s.t. } f_2(x) - s_2 = e_2 \quad (6)$$
$$f_3(x) \quad s_3 = e_3$$
$$\ldots$$
$$f_p(x) - s_p = e_p$$
$$x \in S \text{ and } s_i \in R^+$$

Where δ has chosen small amount which usually it is between 10^{-3} and 10^{-5}, so we have (Xidonas et al., 2010):

$$\max(f_1(x) + \delta.\Sigma_{1=2}^{p} s_i) \quad (7)$$
$$\text{s.t.}$$
$$f_i(x) - s_i - e_i$$
$$1 = 2, \ldots, p$$
$$x \in S \text{ and } s_i \in R^+$$

Where δ is according to above assumption.

Proposition 1: The formulation (7 above) of the ε-constraint method above produces only efficient solutions (it avoids the generation of weak efficient solutions).

Proof. see (Xidonas et al., 2010).

With a stochastic programming approach to multi-portfolio optimization problem, the investor wants to clarify the amount of the return and risk of investing. To achieve this aim, usually ε-constraint method is used where this method keep one of the objective function and considers the other as the constraint.

For instance, in the mean-variance model, expected return is chosen between its minimum and maximum levels. The corresponding optimal variance will be found in consecutive solutions. This approach leads to a discrete denotation of the efficient frontier without producing the whole frontier (Tuncer Şakar and Köksalan, 2012).

5. Results and Discussions

This paper presents a comparison of two methods for multi-portfolio optimization: weighted sum method and ε-constraint method. The first method is based on weighting, by positive coefficients. The second one considers one of the objectives as main function and let others be the constraints.

In this section, some disadvantages and advantages of this method are discussed respectively. First, to illustrate the disadvantages of the ε-constraint method, we start the discussion using one example:

Example. Let the following objective functions and their constraints:

$$\max f_1 - X_1 \quad (8)$$
$$\max f_2 = 3X_1 + 4X_2$$
$$\text{s.t. } X_1 <= 20$$
$$X_2 <= 40$$
$$5X_1 + 4X_2 <- 200$$

According to system above, the feasible region is shown in Figure 1. The red directions denote the two objective functions. The Pareto set is shown in the Figure 1 and result of current ε-constraint method is shown in the Figure 2 as the following:

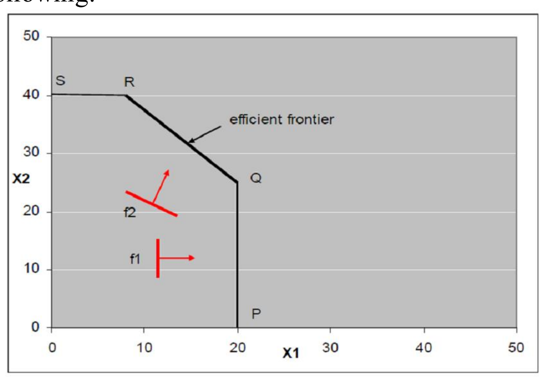

Figure 1: feasible space and objective functions

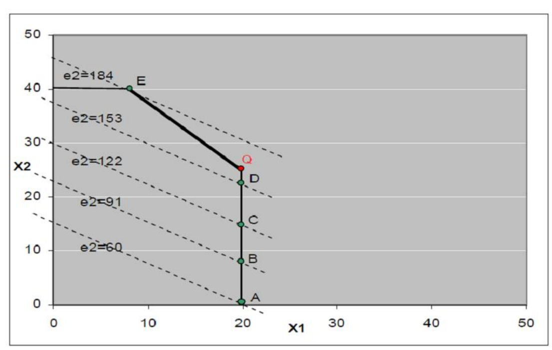

Figure 2: Results of current ε-constraint method

With this method, we can find the points of A, B, C, D, E as the solutions. We see that the point of E dominating the other solution points. So other solutions are weak solutions of this method (Xidonas et al., 2010) (Mavrotas, 2009).

To sum up, some advantages of the ε-constrained method are listed:

1. In the linear case, usually we can use the new version of the problem to find the efficient set with this method.

2. In the multi-objective integer and mixed integer programming cases, we can apply the ε-constraint method while we cannot find them via weighting method (Steuer 1986, Miettinen 1999).

3. Unlike the weighting method, the scaling of the objective functions is not necessary in the ε-constrained method.

4. In this method, the number of the generated efficient solutions can be controlled while it is difficult to control in other method.

5. The range of the objective functions regards to the efficient set and the warranty of effectiveness of the obtained solution are two main notes for the ε-constraint method which is necessary to attend them (Xidonas et al., 2010).

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