Flow of a second grade fluid through curved channel of non-uniform width bounded below by a permeable bed

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Abstract: Creeping flow of an incompressible second grade fluid in a channel of varying width bounded below by a porous bed is solved using three approximate methods based on three different geometrical configurations. Expressions for velocity and pressure gradient are calculated for each method. Weight functions are calculated for all three methods and are evaluated for different values porous layer parameter. It is observed that the weight functions are influenced by porous layer parameter. The obtained results are applied to idealized stenosis geometry and resistance to the flow and shear stress are calculated. It is noticed that as we increase the value of porous layer parameter resistance to the flow in the stenosed region decreases as expected. However with increasing the values of porous layer parameter and non-Newtonian parameter, the shear stress increases in stenosed region. Previous published results are obtained as special case to current study.

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1. Introduction

Porous medium is defined as a material volume consisting of solid matrix with an interconnected void. It is mainly characterized by its porosity, ratio of the void space to the total volume of the medium. Earlier studies in flow in porous media have revealed the Darcy law which relates linearly the flow velocity to the pressure gradient across the porous medium. The porous medium is also characterized by its permeability which is a measure of the flow conductivity in the porous medium. The study of fluid flow in porous media is important for many environmental, industrial and biological problems. Contamination of groundwater, diffusion of tracer particles in cellular bodies, underground oil flow in the petroleum industry and blood flow through capillaries are a few relevant instances where a good understanding of flow in porous media is important. Moreover, there is a wide variety of technical processes that involve fluid dynamics in various branches of process industry. Flow in porous media has been a subject of active research for the last four to five decades. Wiest et al reviewed the mathematical developments and used it to characterize the flow within porous media prior to 1969. He and his co-authors concentrated on natural formations, such as ground water flow through the soil or in underground aquifers.

The flow problems arising from many natural, industrial and biophysical situation, are usually bounded by curved surfaces and thus creating a varying width. In operations involving fluid dynamics, heat and mass transfer, chemical reaction, kinetics and biophysics one can find a huge use of curved configuration. The use of curved channel in some of these applications is required because of geometrical limitations.

A significant amount of interest has been paid to the problem of flow through channels of varying gap and non-uniform pipes filled with porous material. Flows through elastic tubes, blood circulation through capillaries are some of them. The flow of a fluid in an elastic tube, in the absence of porous material, was first approximated by Rashkevsky and later by Morgan. This problem was further investigated by Langlois under several approximations on the geometry of the gap for a Newtonian fluid. Siddiqui et al extended the work of Langlois for a non-Newtonian fluid of second grade. Rudraiah et al further investigated the work of Langlois by assuming that the channel is bounded below by a porous bed. The solution obtained by Rudraiah et al is limited to Newtonian fluids and does not give any information about the non-Newtonian effects, however many industrial and biophysical fluids are non-Newtonian.

The main objective of present work is to get an insight into fluid flow in a varying width channel bounded below by a porous bed for a non-Newtonian fluid of second grade. Three approximate methods have been used for solving the problem. These methods provide an alternate approach to the conventional method of solving a two dimensional problem. A comparison of the three methods has been done by calculating weight functions for each of the methods which are numerically evaluated for different values of $\alpha\sigma$. To study the flow pattern in arteries having stenosis idealized stenosis geometry is considered. In the following sections, the problem is formulated, solved and the obtained results are discussed.

2. Basic Equations

The primary equations that govern the flow of an incompressible second grade fluid in the absence of body forces and thermal effects are:

$$\text{div}\mathbf{V} = 0, \tag{1}$$

$$\rho\dot{\mathbf{V}} = \text{div}\mathbf{T}, \tag{2}$$

where ρ is the constant density, $\mathbf{V}$ is the velocity vector, p is the pressure, dot over $\mathbf{V}$ denotes the material time derivative and $\mathbf{T}$ is the Cauchy stress tensor, which is defined as:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{3}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress modulii, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin-Ericksen tensors respectively, defined as:

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathbf{T}}; \qquad \mathbf{L} = \nabla\mathbf{V}, \tag{4}$$

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathbf{T}}\mathbf{A}_1,$$

and

$$\dot{(\boldsymbol{*})} = (\boldsymbol{*})_t + (\mathbf{V}.\nabla)(\boldsymbol{*}), \tag{5}$$

where $(\boldsymbol{*})_t$ is the partial derivative with respect to t. With the help of equation (3) and (4), momentum equation (2) becomes:

$$\rho\dot{\mathbf{V}} = -\nabla p + \mu\nabla^2\mathbf{V} + (\alpha_1 + \alpha_2)div\mathbf{A}_1^2 + \alpha_1$$
$$+\alpha_1\left[\nabla^2\mathbf{V}_t + \nabla^2(\nabla\times\mathbf{V})\times\mathbf{V} + \nabla\left(\mathbf{V}\cdot\nabla^2\mathbf{V} + \frac{1}{4}|\mathbf{A}_1|^2\right)\right]. \tag{6}$$

3. Problem Formulation

Steady, creeping flow of an incompressible second grade fluid is considered through an infinite horizontal channel which is bounded below by a permeable bed. The flow above the bed called the free flow is governed by the momentum equations and the flow through the bed is governed by the usual Darcy law. Consider cartesian coordinate system such that x-axis is in the direction of the flow and y-axis is perpendicular to it. The porous medium is assumed to be homogeneous and isotropic so that the permeability k is constant. The channel is bounded below by a permeable bed at $y = 0$ and above by a smoothly varying rigid surface:

$$y = h(x) \tag{7}$$

which is continuous and positive for all values of x.

Further more we assume that the porous medium is completely saturated. For plane steady flow we take:

$$\mathbf{V} = [u(x,y), v(x,y)], \tag{8}$$

equation (1) gives:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9}$$

and momentum balance equation(6) in component form yields:

$$-\rho v\Omega = -\frac{\partial\hat{p}}{\partial x} - \mu\frac{\partial\Omega}{\partial y} - \alpha_1 v\nabla^2\Omega, \tag{10}$$

$$\rho u\Omega = -\frac{\partial\hat{p}}{\partial y} + \mu\frac{\partial\Omega}{\partial x} + \alpha_1 u\nabla^2\Omega, \tag{11}$$

where ω is the vorticity defined as:

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{12}$$

and

$$\hat{p}(x,y) = p + \frac{\rho}{2}(u^2 + v^2) - \alpha_1\{u\nabla^2 u + v\nabla^2 v\}$$
$$-\frac{1}{4}(3\alpha_1 + 2\alpha_2)|A_1^2|. \tag{13}$$

Since for creeping flow the convective part of momentum equation is negligibly small, equations (10) and (11) reduce to:

$$\frac{\partial\hat{p}}{\partial x} = -\mu\frac{\partial\Omega}{\partial y} - \alpha_1 v\nabla^2\Omega, \tag{14}$$

$$\frac{\partial\hat{p}}{\partial y} = \mu\frac{\partial\Omega}{\partial x} + \alpha_1 u\nabla^2\Omega, \tag{15}$$

where

$$\hat{p}(x,y) = p - \alpha_1\{u\nabla^2 u + v\nabla^2 v\} - \frac{1}{4}(3\alpha_1 + 2\alpha_2)|A_1^2|, \tag{16}$$

and

$$|A_1^2| = 8\left|\frac{\partial u}{\partial x}\right|^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2. \tag{17}$$

The boundary conditions suitable for this problem are the no slip conditions at the upper rigid surface and a slip condition at the bounding surface similar to the one postulated by Beavers and Joseph (BJ) [7], i.e.

(i) No – slip conditions:

$$u(x,y) = v(x,y) = 0, \quad \text{at} \quad y = h(x) \quad \text{for all x} \tag{18}$$

(ii) BJ slip condition:

$$\frac{du}{dy} = \frac{\alpha(u_B - Q)}{\sqrt{k}} \quad \text{at} \quad y = 0 \tag{19}$$

$$v = v_m$$

where $Q = -\frac{k}{\mu}\frac{\partial p}{\partial x}$ is the Darcy velocity at $y = 0$,

u_B is the slip velocity, k is the permeability of the porous material, α is the dimensionless constant and v_m is the normal component of Darcy velocity. Since the media is completely saturated, we assume that $v_m = 0$ and hence the boundary condition on normal component of velocity at the nominal surface is $v = 0$ at $y = 0$. Eliminating the pressure between equations (14) and (15) we have a third order differential equation for u and v. Hence in addition to the no-slip boundary condition (18) and BJ slip conditions (19) we need one more boundary condition. This boundary condition is obtained by calculating the mass flux across the channel, which has to be constant at all cross-sections of the channel for an incompressible second grade fluid. Hence the required third boundary condition is:

$$(iii) \int_{-h(x)}^{h(x)} u dy = m, \qquad (20)$$

which is a constant for all x.

4. Solution of the Problem

To find the solution of the problems three approximate methods depending upon the three physical situations are used which are discussed in the following section. Equations (9), (14) and (15) are three partial differential equations for three unknown functions u, v and p. Once the velocity field is determined the pressure field (16) can be calculated by integrating equations (14) and (15). When suitable restrictions are placed upon $h(x)$, the boundary value problem represented by equations (9), (14) and (15) together with the boundary conditions (18)-(20) is precisely of the sort in which it is helpful to look at the local picture of flow.

4.1 Method 1: *Negligible Wall Slope:*

The function $h(x)$ is assumed in such a way that its first derivative $h'(x)$ is small everywhere compared with unity. Then it is reasonable to assume that at each value of x the component of velocity and pressure gradient are approximately equal to those obtaining in a channel of uniform width. Thus in order to satisfy the no-slip boundary condition, v has to be zero throughout the channel. This approximation leads to Poiseuille flow in a channel of uniform width with BJ conditions. The differential equations (14) and (15) then reduce to:

$$\frac{\partial \hat{p}}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, \quad v = 0, \qquad (21)$$

$$\frac{\partial \hat{p}}{\partial y} = 0, \qquad (22)$$

with

$$\hat{p} = p - \alpha_1 u \frac{\partial^2 u}{\partial y^2} - \frac{1}{2}(3\alpha_1 + 2\alpha_2)(\frac{\partial u}{\partial y})^2. \qquad (23)$$

Eliminating p from (21) and (22), we get:

$$\frac{\partial^3 u}{\partial y^3} = 0. \qquad (24)$$

Solving equation (24) using boundary conditions (18) to (20) we find that:

$$u = \frac{3m}{2h^3}(h^2 - y^2) - \frac{\alpha\sigma(3m - 2hQ)}{2h^3(4 + \alpha\sigma)}(3y^2 - 4hy + h^2), \qquad (25)$$

where

$$m = \frac{h^3}{12\mu} \frac{6\alpha + \sigma(4 + \alpha\sigma)}{\sigma(1 + \alpha\sigma)} \left(-\frac{\partial p}{\partial x} \right) \qquad (26)$$

The expression for pressure distribution from equations (16) with the help of equations (14)-(15), (17) and (25)-(26) becomes:

$$p = 3\mu m \int_x^C \frac{dx}{h^3} + 3\mu\alpha \int_x^C \frac{\sigma(3m - 2hQ)}{h^3(4 + \alpha\sigma)} - \frac{\alpha_1}{2h^6(4 + \alpha\sigma)^2}$$
$$\Big[9(4 + \alpha\sigma)(4m^2(1 + \alpha\sigma) - 2\alpha\sigma hQ)(h^2 - y^2) -$$
$$3\alpha\sigma(3m - 2hQ)(4m(1 + \alpha\sigma) - 2h\alpha\sigma Q)(3y^2 - 4hy + h^2)\Big]$$
$$+ \frac{(3\alpha_1 + 2\alpha_2)}{2h^6(4 + \alpha\sigma)^2}\Big[9m^2(4 + \alpha\sigma)^2$$
$$+ \alpha\sigma(3m - 2hQ)(6y - 4h)\left\{3m(4 + \alpha\sigma)y + \frac{1}{4}\alpha\sigma(3m - 2hQ)(6y - 4h)\right\}\Big] \qquad (27)$$

Equations (25)-(27) reduce to

- solution obtained by Langlois as $\sigma \to \infty$ and by setting $\alpha_1 = \alpha_2 = 0$.
- solution obtained by Rudraiah et al by setting $\alpha_1 = \alpha_2 = 0$.
- solution obtained by Siddiqui et al as $\sigma \to \infty$.

Direct substitution of velocity profile (25) into (18)-(20) reveals that the boundary conditions are satisfied. The equations (9), (14)-(15) are satisfied by velocity profile (25) and pressure distribution (27) provided we neglect terms involving $h'(x)$ and $h(x)h''(x)$. The validity of this approach therefore requires that:

$$|h'(x)| << 1, \qquad (28a)$$

$$|h(x)h'(x)| << 1, \qquad (28b)$$

$$|h^2(x)h'(x)| << 1. \qquad (28c)$$

for all x

4.2 Method 2: *Negligible Wall Curvature:*

We can remove the restriction on wall slope by assuming that the channel width $h(x)$ is a linear function of x. Depending upon the sign of $h'(x)$ we have different geometrical situations of the channel. If $h'(x)$ is positive, the channel is approximated by a divergent wedge with a source of flux Q at its vortex. If $h'(x)$ is negative, the wedge is convergent with a sink at its vortex. This analysis is carried out assuming $h'(x)$ as positive, and similar results can be obtained when $h'(x)$ is negative.

The equations of motion for the fully developed creeping flow of a second grade fluid in plane polar coordinates are:

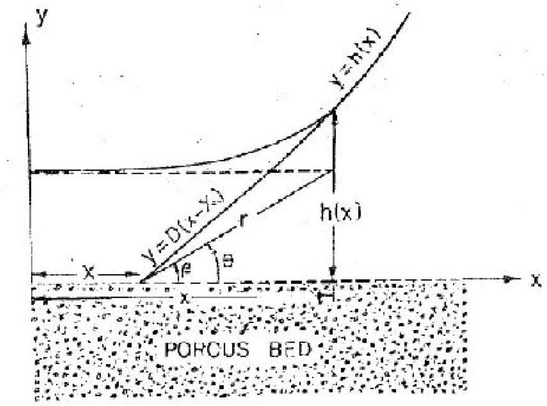

Figure 2: Wedge flow geometry.

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial \theta}=0, \tag{28}$$

$$\frac{\partial \hat{p}}{\partial r}=\mu\left[\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right]-\alpha_1\, v\,\nabla^2\Omega, \tag{29}$$

$$\frac{\partial \hat{p}}{\partial \theta}=\mu\, r\left[\nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}\right]+\alpha_1\, r\, u\,\nabla^2\Omega, \tag{30}$$

where $\hat{p}$ is modified pressure and is defined as:

$$\hat{p}=p-\left[\begin{array}{l}\alpha_1\left\{u\left(\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)+v\left(\begin{array}{l}\nabla^2 v-\frac{v}{r^2}\\ +\frac{2}{r^2}\frac{\partial u}{\partial \theta}\end{array}\right)\right\}\\ +\frac{(3\alpha_1+2\alpha_2)}{4}\,|A_1^2|\end{array}\right], \tag{31}$$

and

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \theta} \tag{32}$$

$$\Omega=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{33}$$

If we assume the radial component of velocity to be of the form $u=\frac{f(\theta)}{r}$, then continuity equation (28) shows that $\frac{\partial v}{\partial \theta}=0$. Since the media is saturated, to satisfy the no-slip boundary conditions, v has to be zero throughout the fluid. Thus no slip boundary conditions suitable for this problem are:

$$u(r,\beta)=v(r,\beta)=0 \tag{34}$$

The slip boundary condition, at the normal surface is,

$$f'(\theta)=\alpha\sigma_0(f_B-Q_1) \quad \text{when} \quad \theta=0 \tag{35}$$

where

$$f_B=\frac{f(0)}{r_0}=u_B,\; Q_1=Qr_0,\; Q=-\frac{k}{\mu}\frac{\partial p}{\partial r}$$

$$\sigma_0=\frac{r_0}{\sqrt{k}}\text{, i.e. } h\sigma_0=r_0\sigma,\; r_0$$

is the the value of r when $\theta=0$.

Further

$$\int_0^\beta ru\,d\theta=m \qquad \text{i.e.} \qquad \int_0^\beta f(\theta)\,d\theta=m \tag{36}$$

$$u=\frac{f(\theta)}{r},\quad v=0, \tag{37}$$

Equations (29) to (30) become

$$\frac{\partial \hat{p}}{\partial r}=\frac{f''(\theta)}{r^3}, \tag{38}$$

$$\frac{1}{r}\frac{\partial \hat{p}}{\partial \theta}=\frac{2f'(\theta)}{r^3}-\frac{\alpha_1}{r^5}\left[4f(\theta)f'(\theta)+f(\theta)f'''(\theta)\right], \tag{39}$$

where

$$\hat{p}=p-\alpha_1\frac{f(\theta)f''(\theta)}{r^4}-\frac{(3\alpha_1+2\alpha_2)}{2}\left(4f(\theta)^2+f'(\theta)^2\right) \tag{40}$$

Eliminating pressure gradient between (38)-(39) by cross differentiation, we find that

$$\frac{1}{r^3}(f'''+4f')-\frac{4\alpha_1}{r^5}\left(ff'''+4ff'\right)=0, \tag{41}$$

From equation (41) we have either

$$\frac{1}{r^3}(f'''+4f')=0, \tag{42}$$

or

$$\frac{4\alpha_1}{r^5}\left(ff'''+4ff'\right)=0. \tag{43}$$

Solving equation (42) using the boundary conditions (35)-(36), we obtain:

$$f=m\gamma(\cos 2\beta-\cos 2\theta)+A\gamma\left\{\begin{array}{l}\sin\beta\cos\theta\\ -\beta\cos(\theta-\beta)\end{array}\right\}\sin(\theta-\beta) \tag{44}$$

It can be easily verified that equation (44) satisfy equation (43). Tanner [8] had shown that any plane creeping Newtonian velocity field is also a solution for second grade fluid under identical velocity boundary conditions. Following him we can conclude that equation (44) is also a solution of equation (43). Using equation (44) into equation (37), we obtain :

$$u=\frac{m\gamma}{r}(\cos 2\beta-\cos 2\theta)+\frac{A\gamma}{r}\begin{Bmatrix}\sin\beta\cos\theta\\ -\beta\cos(\theta-\beta)\end{Bmatrix}\sin(\theta-\beta),$$

$$v=0, \tag{45}$$

$$\begin{aligned}\frac{\partial p}{\partial r}&=-\frac{2m\gamma\mu}{r^3}\cos(2\theta)-\frac{2A\gamma\mu}{r^3}(\sin(2\theta-\beta)\sin(\beta)-\beta\sin 2(\theta-\beta))\\ &-\frac{2(3\alpha_1+2\alpha_2)\gamma^2}{r^5}[4m^2(\cos^2(\theta)\sin^2(\theta)+(\sin^2(\beta)-\sin^2(\theta))^2)\\ &-Am(8\cos^2(\theta)\sin^4(\beta)-2\beta(\cos^2(2\beta)\sin(2\theta)+\sin(2\beta))\\ &+\frac{1}{2}(2\beta\cos(2\theta)+\sin(2\theta))\sin(4\beta))+\frac{A^2}{8}\begin{pmatrix}(7+8\beta^2+\cos(4\beta))\\ -2\cos(2\theta)\sin^3(\beta)(2\beta\cos(\beta)\end{pmatrix}\\ &-\sin(\beta))-\cos(2\beta)(1-2\beta\sin(2\theta)\sin(\beta)^2)-(\beta+\sin(2\theta)\sin(\beta)^2)\sin(2\beta))]\\ &+\frac{4\alpha_1\gamma^2}{r^5}[2m^2((\cos^2(\theta)-\sin^2(\theta))(\sin^2(\beta)-\sin^2(\theta))) \\ &+\frac{Am}{2}(\sin(\theta-\beta)(2(\cos(\theta)-\cos(\theta-2\beta)+\begin{matrix}2\beta(\sin(\theta)-\sin(3\theta)+\sin(\theta-2\beta)))\sin(\beta)\\ -4\cos(3\theta)(\beta\cos(\beta)-\sin(\beta))\end{matrix}))\\ &+\frac{A^2}{2}((\cos(2\theta)-\begin{matrix}\cos 2(\theta-\beta)+2\beta\sin 2(\theta-\beta))\sin(\theta-\beta)(2\beta\cos(\theta-\beta)\\ +\sin(\theta-\beta)-\sin(\theta+\beta))\end{matrix})],\end{aligned}\tag{46}$$

$$\begin{aligned}\frac{\partial p}{\partial\theta}&=-\frac{2m\gamma\mu}{r^2}\sin(2\theta)+\frac{2A\gamma\mu}{r^2}(\cos(2\theta-\beta)\sin(\beta)-\beta\cos 2(\theta-\beta))\\ &+\frac{(3\alpha_1+2\alpha_2)\gamma^2}{2r^4}[8m^2(2(\sin^2(\beta)-\sin^2(\theta))-(\cos(\theta)^2-\sin(\theta)^2))\\ &\cos(\theta)\sin(\theta)+Am(4\beta\cos 2(\theta-\beta)\cos(2\beta)+8\sin(2\theta)\sin^4(\beta)\\ &-\cos(2\theta)\sin(4\beta))+A^2(4\sin(\beta)^2(\beta\cos 2(\theta-\beta)-\cos(2\theta-\beta)\sin(\beta)))]\\ &+\frac{\alpha_1\gamma^2}{r^4}[4m^2((\cos^2(\theta)-\sin^2(\theta))+2(\sin^2(\beta)-\sin^2(\theta)))\cos(\theta)\sin(\theta)\\ &-\frac{Am}{2}((2\beta\cos(2\theta)+2\beta\cos(2\theta-4\beta)-8\beta\cos(4\theta-2\beta)+3\sin(2\theta)\\ &+4\sin(4\theta)+\sin(2\theta-4\beta)-2\sin 2(\theta-\beta)-4\sin(4\theta-2\beta)\\ &-2\sin 2(\theta+\beta)))-\frac{A^2}{2}\begin{pmatrix}(2\beta\cos 2(\theta-2\beta)-\sin(2\theta)+\sin 2(\theta-\beta))\\ \left(1+2\cos(2\theta)-2\cos 2(\theta-2\beta)-\begin{matrix}\cos(2\beta)\\ +4\beta\sin 2(\theta-\beta)\end{matrix}\right)\end{pmatrix}],\end{aligned}\tag{47}$$

and

$$\begin{aligned}p&=\frac{m\gamma\mu}{r^2}\cos(2\theta)+\frac{A\gamma\mu}{r^2}(\sin(2\theta-\beta)\sin(\beta)-\beta\sin 2(\theta-\beta))\\ &+\frac{(3\alpha_1+2\alpha_2)\gamma^2}{2r^4}[4m^2(\cos^2(\theta)\sin^2(\theta)+(\sin^2(\beta)-\sin^2(\theta))^2)\\ &-Am(8\cos^2(\theta)\sin^4(\beta)-2\beta(\cos^2(2\beta)\sin(2\theta)+\sin(2\beta))\\ &+\frac{1}{2}(2\beta\cos(2\theta)+\sin(2\theta))\sin(4\beta))+\frac{A^2}{8}((7+8\beta^2+\cos(4\beta))\\ &-2\cos(2\theta)\sin^3(\beta)(2\beta\cos(\beta)-\begin{matrix}\sin(\beta))-\cos(2\beta)\begin{pmatrix}1-2\beta\sin(2\theta)\\ \sin(\beta)^2\end{pmatrix}\\ -(\beta+\sin(2\theta)\sin(\beta)^2)\sin(2\beta)\end{matrix})]\\ &-\frac{\alpha_1\gamma^2}{r^4}[2m^2((\cos^2(\theta)-\sin^2(\theta))(\sin^2(\beta)-\sin^2(\theta)))\\ &+\frac{Am}{2}(\sin(\theta-\beta)(2(\cos(\theta)-\cos(\theta-2\beta)\\ &+2\beta(\sin(\theta)-\sin(3\theta)+\sin(\theta-2\beta)))\sin(\beta)-4\cos(3\theta)(\beta\cos(\beta)-\sin(\beta))))\\ &+\frac{A^2}{2}((\cos(2\theta)-\begin{matrix}\cos 2(\theta-\beta)+2\beta\sin 2(\theta-\beta))\sin(\theta-\beta)(2\beta\cos(\theta-\beta)\\ +\sin(\theta-\beta)-\sin(\theta+\beta))\end{matrix})]+C,\end{aligned}\tag{48}$$

where

$$\gamma=\frac{1}{\sin\beta\cos\beta-\beta+2\beta\sin^2\beta}, \tag{49}$$

$$A=\frac{\alpha\sigma_0(4m\sin^2\beta-Q_1\sin 2\beta+2Q_1\cos 2\beta)}{(1-\alpha\sigma_0\beta)\sin 2\beta-2\beta\cos 2\beta+2\alpha\sigma_0\sin^2\beta}. \tag{50}$$

Equations (45)-(50) reduce to:

- solution obtained by Langlois as $\sigma_0\to\infty$ and by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Rudraiah et al by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Siddiqui et al as $\sigma_0\to\infty$.

Equations (45)-(50) satisfy the boundary conditions and differential equations (29)-(30). In order to use these results in varying width channel problem, we convert them to cartesian coordinates, with the following notation indicated in figure 2.

$$\begin{aligned}&u=u_r\cos\theta,\quad \cos\theta=\frac{(x-X)}{r}=\frac{h}{Dr},\\ &v=u_r\sin\theta,\quad \sin\theta=\frac{y}{r},\\ &\sin\alpha=\frac{D}{\sqrt{1+D^2}},\quad \cos\alpha=\frac{1}{\sqrt{1+D^2}},\\ &\tan\alpha=\frac{h}{x-X}=D,\quad \tan\theta=\frac{y}{x-X}=\frac{Dy}{h},\\ &r=\sqrt{(x-X)^2+y^2}=\frac{\sqrt{h^2+D^2y^2}}{D},\\ &(x-X)=\frac{h}{D}\end{aligned}\tag{51}$$

The components of pressure gradient are given by:

$$\begin{aligned}\frac{\partial p}{\partial x}&=\frac{h}{Dr}\frac{\partial p}{\partial r}-\frac{y}{r^2}\frac{\partial p}{\partial\theta},\\ \frac{\partial p}{\partial y}&=\frac{y}{r}\frac{\partial p}{\partial r}+\frac{h}{Dr^2}\frac{\partial p}{\partial\theta}.\end{aligned}\tag{52}$$

Equations (45)-(50) using equations (50)-(51) take the following form:

$$\begin{aligned}u&=\frac{2mD^3h(h^2-y^2)}{E(h^2+D^2y^2)^2}+\frac{A_0D^2h}{E}\\ &\frac{[Dh-(h+D^2y)\beta](y-h)}{(h^2+D^2y^2)^2},\end{aligned}\tag{53}$$

$$\begin{aligned}v&=\frac{2mD^4y(h^2-y^2)}{E(h^2+D^2y^2)^2}+\frac{A_0D^3y}{E}\\ &\frac{[Dh-(h+D^2y)\beta](y-h)}{(h^2+D^2y^2)^2},\end{aligned}\tag{54}$$

$$\frac{\partial p}{\partial x}=-\frac{4\mu m(1+D^2)D^3h}{E(h^2+D^2y^2)^3}(h^2-3D^2y^2)$$
$$-\frac{2A_0\mu D^4}{E}\left[Ey\frac{3h^2-D^2y^2}{(h^2+D^2y^2)^3}-(D-2\beta)h\frac{h^2-3D^2y^2}{(h^2+D^2y^2)^3}\right]$$
$$-\frac{4D^5(3\alpha_1+2\alpha_2)}{E^2(h^2+D^2y^2)^4}(m^2D^2h(2D^2h^2+3y^2-D^4y^2)$$
$$-\frac{A_0mD}{2}(D(3h^2y+D^4(4h-y)y^2+D^2(8h^3$$
$$-\frac{4D^5(3\alpha_1+2\alpha_2)}{E^2(h^2+D^2y^2)^4}(m^2D^2h(2D^2h^2+3y^2-D^4y^2)$$
$$3h^2y+y^3))$$
$$-\frac{A_0mD}{2}\left(D(3h^2y+D^4(4h-y)y^2+D^2(8h^3-\begin{pmatrix}8D^2h^3\\-(D-3)(D-1)^2h^2y\\+4D^2(1+D)hy^2\\+(D-1)^2D^2(1+D)y^3\end{pmatrix}\beta\right)$$
$$+\frac{A_0^2}{2(h^2+D^2y^2)}\left(D^2\begin{pmatrix}(1+4D^2)h^5-D(1-(D-7)D)h^4y+2(D-1)^2D^2h^3y^2\\+D^4\left(1+2\begin{pmatrix}D-2)D)hy^4-D^5(D+D^2-1)y^5)\\-(2Dh((1+3D^2)h^4-4D^2(2+D^2)h^3y+2D^2(1+5D^2)\end{pmatrix}\right.\end{pmatrix}\right.$$
$$\times h^2y^2-4D^4hy^3-D^4(D^2-1)y^4))\beta+\left(h(h^2+D^2h^2-4D^2hy+D^2y^2+D^4y^2)^2\right)\beta^2))$$
$$+\frac{4\alpha_1D^6}{E^2(h^2+D^2y^2)^5}\Big(Dm^2(1+D^2)h(2h^4-3h^2y^2-5D^2h^2y^2+5D^2y^4+D^4y^4)-\frac{A_0m}{2}(D(2(2$$
$$+D+3D^2)h^5-(5+D(2+5D))h^4y+2D^2(D-11D^2-12)h^3y^2+2D^2(12+D(13D-1))h^2y^3$$
$$+4D^4(1+D^2)hy^4-D^4(3+D^2)y^5)-(4(1+3D^2)h^5+(D-1)(5+3D)(1+3D^2)h^4y-4D^2$$
$$\times(7+D+8D^2)h^3y^2-4(D-1)D^2(2+D)(3+2D^2)h^2y^3+4D^4(8+D(5D-1))hy^4-D^4(3$$
$$-2D-2D^3+D^4)y^5)\beta)+\frac{A_0^2D}{4}(D(8Dh^5-(1+29D)h^4y-2D(D+9D^2-12)h^3y^2+32D^3h^2y^3$$
$$+2D^3(1+D)(3D-4)hy^4+(1-3D)D^4y^5)-(24Dh^5+(D(D+29D^2-69)-1)h^4y-4D(D$$
$$+29D^2-12)h^3y^2+16D^3(7-2D^2)h^2y^3+4D^3(D(13D-1)-4)hy^4+D^4(1+D(D(3D-1)$$
$$-11))y^5)\beta-((8(h-y)(h+D^2y)(2h^3+3(D^2-1)h^2y-6D^2hy^2-D^2(D^2-1)y^3))\beta^2)\Big), \tag{55}$$

$$\frac{\partial p}{\partial y}=-\frac{4\mu m(1+D^2)D^4y}{E(h^2+D^2y^2)^3}(h^2-3D^2y^2)+\frac{2A_0\mu D^3}{E}$$
$$\left[Eh\frac{h^2-3D^2y^2}{(h^2+D^2y^2)^3}-D^2(D-2\beta)y\frac{3h^2-D^2y^2}{(h^2+D^2y^2)^3}\right]$$
$$-\frac{4(3\alpha_1+2\alpha_2)D^5}{E^2(h^2+D^2y^2)^4}(Dy(3D^4h^2+2D^2y^2-h^2)$$
$$+\frac{A_0m}{2}(Dh((D^2-1)h^2+4D^4hy-3D^2(D^2-1)y^2)+$$
$$\left(\begin{pmatrix}(D-1)^2(1+D)h^3+4D^2(1+D(2D-1))h^2y\\-(D-3)(D-1)^2D^2hy^2+8D^4y^3\end{pmatrix}\beta\right)$$
$$-A_0^2D(8D(D+D^2-1)h^3+(7+D+7D^2+33D^3-6D^4)h^2y$$
$$+4D^3(D+2D^2-2)hy^2+D^2(7+D+7D^2+D^3+6D^4)y^3$$
$$+(2Dy((1+3D^2)h^2+2(D-1)D^2hy-D^2(D^2-1)y^2))\beta$$
$$-(8(1+D^2)^2y(h^2+D^2y^2))\beta^2))+\frac{4\alpha_1D^5}{E^2(h^2+D^2y^2)^5}(m^2D(1+D^2)$$
$$y(h^4+5D^2h^4-5D^2h^2y^2-3D^4h^2y^2+2D^4y^4)$$
$$+\frac{A_0m}{2}(D((1+3D^2)h^5+2D^2(6+D+9D^2)h^4y$$
$$-2D^2(8+D+7D^2)h^3y^2-2D^4(12+D(7D-1))$$
$$\times h^2y^3+D^4(15-(2-15D)D)hy^4-4D^6y^5)$$
$$+((D-1)(1+D(D(3+D))-1)h^5-16D^2(1+2D^2)h^4y$$
$$-4(D-1)D^24+D(1+2D(3+D)))h^3y^2$$
$$+4D^4(9+D+6D^2)h^2y^3+(D-1)D^4(1+D)$$
$$(15+D(7D-2))hy^4-4D^6(3+D(2D-1))y^5)\beta)$$
$$-\frac{A_0^2D}{4}((1+5D)h^4+2D(D+9D^2-4)h^3y-48D^3h^2y^2$$
$$-2D^3(D(7D-1)+12)hy^3-(1-11D)D^4y^4$$
$$+((D(D+5D^2-13)-1)h^5-4D(D+19D^2-4)h^4y$$
$$-16D^3(3D^2-8)h^3y^2-4D^3(12+D-27D^2)h^2y^3$$
$$-D^4(D(51+D-11D^2)-1)hy^4-8D^7y^5)\beta$$
$$+(8(h-y)(h+D^2y)(-(D^2-1)h^3+6D^2h^2y+3D^2(D^2-1)hy^2-2D^4y^3))\beta^2)), \tag{56}$$

where

$$E=D-\beta(1-D^2) \text{ and } A_0=\frac{\alpha\sigma_0(2mD^2-Q_1E)}{E+\alpha\sigma_0D(D-\beta)}.$$

Upon checking by direct substitution, we find that the velocity components given by equations (52) and (53) satisfy the boundary conditions (18)-(20). We also find that equationa (52)-(56) satisfy the differential equations (9), (14) and (15), provided that:

$$|\,h(x)h''(x)\,|<<1, \tag{57}$$

$$|\,h(x)^2h'''(x)\,|<<1. \tag{58}$$

If condition (57) is satisfied, it can be verified that

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy, \tag{59}$$

is an exact differential when $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are given by equations (55) and (56). Therefore:

$$p=\int_x^C\frac{\partial p}{\partial x}dx+\int_0^y\frac{\partial p}{\partial y}dy, \tag{60}$$

where C is a constant of integration $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are given by equations (55) to (56).

Equations (53)-(56) reduce to:

- solution obtained by Langlois as $\alpha\sigma_0\to\infty$ and by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Rudraiah et al by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Siddiqui et al as $\alpha\sigma_0\to\infty$.

4.3 Method 3: *Wall Slope Expanded in Power Series:*

Method 2 discussed above gave cumbersome results even for analytically simple form of $h(x)$. It may also happen that the function $h(x)$ satisfy equations (57) for small $h'(x)$ but not negligible. So we can modify the condition:

$$|\,D^n\,|=|\,h'(x)^n\,|<<1, \tag{61}$$

which is satisfied for some positive integer n. For $n>1$ we expand the results of method 2 in power series in D, neglecting the terms of $n=3$ or higher order in D. For this we proceed as follows: Expanding $\tan^{-1}(D)$ in power series of D, we have:

$$\tan^{-1}(D)=D-\frac{1}{3}D^3+\frac{1}{5}D^5+O(D^7). \tag{62}$$

The function $E=D-\beta(1-D^2)$ when expanded in powers of D, gives:

$$E=\frac{4}{3}D^3\left[1-\frac{2}{5}D^2+O(D^4)\right], \tag{63}$$

which after some manipulation, becomes:

$$\frac{D^3}{E}=\frac{3}{4}\left[1+\frac{2}{5}D^2+O(D^4)\right]. \qquad (64)$$

Similarly the function E^2 gives:

$$\frac{D^6}{E^2}=\frac{9}{16}\left[1+\frac{4}{5}D^2+O(D^4)\right]. \qquad (65)$$

The expressions for the velocity components and pressure, expanding in powers of D and neglecting the third or higher order in D, become:

$$u=\frac{3m}{4h}\left[1-\left(\frac{y}{h}\right)^2\right]\left[1-2D^2\left(\frac{y}{h}\right)^2+\frac{2}{5}D^2\right]-\frac{9\alpha\sigma_0 D}{8h}\left[\frac{1}{3}-\frac{4}{3}\left(\frac{y}{h}\right)+\left(\frac{y}{h}\right)^2\right]A_{\Diamond}, \qquad (66)$$

$$v=\frac{3mD}{4h}\left(\frac{y}{h}\right)\left[1-\left(\frac{y}{h}\right)^2\right]-\frac{9\alpha\sigma_0 D}{8h}\left[\frac{1}{3}-\frac{4}{3}\left(\frac{y}{h}\right)+\left(\frac{y}{h}\right)^2\right], \qquad (67)$$

$$\frac{\partial p}{\partial x}=\frac{-3\mu m}{2h^3}\left[1-6D^2\left(\frac{y}{h}\right)^2+\frac{7}{5}D^2\right]-\frac{9\mu\alpha\sigma_0 D}{4h^3}\left[m(1-\frac{\alpha\sigma_0}{4}D)-\frac{2}{3}Q_1 D\right]A_{\Diamond} \qquad (68)$$
$$-4(3\alpha_1+2\alpha_2)\left[\frac{27m^2 D}{16h^5}\left(\frac{y}{h}\right)^2-\frac{63A_{\Diamond}mD^2}{32h^5}\left(\frac{y}{h}\right)-\frac{9A_{\Diamond}^2 D^2}{32h^5}\left(\frac{y}{h}\right)\right]$$
$$+4\alpha_1\left[\frac{9m^2 D(1+D^2)}{16h^5}\left(2-3\left(\frac{y}{h}\right)^2\right)-\frac{9A_{\Diamond}mD^2}{16h^5}\left(1-4\left(\frac{y}{h}\right)\right)\right],$$

$$\frac{\partial p}{\partial y}=\frac{-9\mu mD}{2h^3}\left(\frac{y}{h}\right)-\frac{9\mu\alpha\sigma_0 D^2}{4h^3}\left[\frac{4}{3}-3\left(\frac{y}{h}\right)\right]-4(3\alpha_1+2\alpha_2)\left[\frac{-9}{16h^5}\left(\frac{y}{h}\right)\left(1+D^2\left(\frac{4}{5}-6\left(\frac{y}{h}\right)^2\right)\right)\right.$$
$$\left.-\frac{9A_{\Diamond}Dm}{32h^5}\left(1-4D\left(\frac{y}{h}\right)\right)+\frac{9A_{\Diamond}^2 D}{16h^5}\left(8+\left(\frac{y}{h}\right)+\frac{7}{D}\left(\frac{y}{h}\right)-D\left(8-\frac{93}{5}\left(\frac{y}{h}\right)+21\left(\frac{y}{h}\right)^3\right)\right)\right]$$
$$+4\alpha_1\left[\frac{9m^2(1+D^2)}{16h^5}\left(\frac{y}{h}\right)\left(1+D^2\left(\frac{29}{5}-10\left(\frac{y}{h}\right)^2\right)\right)+\frac{9A_{\Diamond}mD}{32h^5}\left(2-D\left(1+4\left(\frac{y}{h}\right)\right)\right)\right], \qquad (69)$$

and the equation for the pressure distribution becomes :

$$p=\frac{3\mu m}{4}\left[-2\int_x^C\frac{(1+\frac{7}{5}D^2)}{h^3}dx-\frac{3D}{h^2}\left(\frac{y}{h}\right)^2\right]+\frac{9\mu\alpha}{4}\left[\int_x^C\frac{\sigma_0 D}{h^3}\left\{m(1\frac{\alpha\sigma_0 D}{4})-\frac{2}{3}Q_1 D\right\}Dx\right.$$
$$\left.+\frac{\sigma_0 mD^2}{h^2}\left(\frac{4}{3}-\frac{3}{2}\frac{y}{h}\right)\right]+4(3\alpha_1+2\alpha_2)y\left[\frac{9y}{16h^6}\left(1+D^2\left(\frac{4}{5}-\frac{6y^2}{h^2}\right)\right)+\frac{9A_{\Diamond}Dm}{32h^5}\left(1-\frac{4Dy}{h}\right)\right.$$
$$\left.-\frac{9A_{\Diamond}^2 D}{16h^5}\left(8+\frac{y}{h}+\frac{7y}{Dh}-D\left(8-\frac{93y}{5h}+\frac{21y^3}{h^3}\right)\right)\right]+\frac{9\alpha_1}{4}\left[2m^2\int_x^C\frac{D(1+D^2)}{h^5}+y\left(\frac{(1+D^2)m^2 y}{h^6}\right.\right.$$
$$\left.\left.\times\left(1+D^2\left(\frac{29}{5}-\frac{10y^2}{h^2}\right)\right)\frac{A_{\Diamond}Dm}{2h^5}\left(2-D\left(1+\frac{4y}{h}\right)\right)\right)\right],$$

(70)

where $A_{\Diamond}=\left[m(1-\frac{\alpha\sigma_0}{4}D)-\frac{2}{3}Q_1 D\right]$.

Equations (66)-(70) reduce to:

- solution obtained by Langlois as $\sigma_0\to\infty$ and by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Rudraiah et al by setting $\alpha_1=\alpha_2=0$.
- solution obtained by Siddiqui et al as $\sigma_0\to\infty$.

4.4 *Comparison of the Three Methods:*

In order to have an idea of applicability of the first and third approximate methods, in this section a calculation has been made using all the three methods.

Average pressure gradient across the channel is:

$$\Delta p=\frac{-1}{h(x)}\int_0^{h(x)}\frac{\partial p}{\partial x}dy, \qquad (71)$$

Δ is calculated in the neighborhood of a given value of x using the three methods mentioned above and the weight functions are compared. Δp is found to be

$$\Delta p=\frac{3\mu m}{2h^3}F_i(D), \quad \text{for} \quad i=1,2,3, \qquad (72)$$

where $F_i(D)$, for $i=1,2,3$ are given by:

$$F_1(D)=\left(1+\frac{\alpha\sigma_0(3-2Q_0)}{(4\sigma_1+\alpha\sigma_0)}\right)+\frac{(3\lambda_1+2\lambda_2)D}{(4\sigma_1+\alpha\sigma_0)^2 h^4}$$
$$\left(27m(4\sigma_1+\alpha\sigma_0)^2-9(4\sigma_1-3\alpha\sigma_0)h^2 m\alpha\sigma_0\right. \qquad (37)$$
$$\left.+24m(\sigma_1-2\alpha\sigma_0)h^2 Q_0\alpha\sigma_0 20h^3 Q_0\alpha{\sigma_0}^2\right)$$
$$-\frac{4\alpha_1 D(3m^2(\sigma_1+\alpha\sigma_0)-\alpha\sigma_0 Q_0)}{(4\sigma_1+\alpha\sigma_0)h^2},$$

(73)

$$F_2(D)=\frac{4D^3}{3(1+D^2)E}+\frac{A_0}{m}\frac{D^3}{3(1+D^2)E}\left[D+(1+D^2)\tan^{-1}D\right] \qquad (36)$$
$$+\frac{(3\lambda_1+2\lambda_2)D^4}{3E^2h^2\mu}\left[\frac{m}{3}\left(D(3-11D^2\right.\right.$$
$$\left.-15D^4+9D^6)-3\left((1+D^2)^2(1+3D^4)\right)\tan^{-1}D\right)$$
$$+\frac{A_0 D}{3(1+D^2)}\left(2D^2(1+D^2)(9+24D^2+13D^4)\right.$$
$$-\left(D\left(\begin{matrix}15-45D+92D^2-43D^3\\+18D^4-15D^5-60D^6-D^7-33D^8\end{matrix}\right)+3(1+D^2)^3\right.$$
$$\left.\times(1+D+10D^2)\tan^{-1}D\right)\tan^{-1}D\Big)-\frac{{A_0}^2}{24m(1+D^2)^2}$$
$$\left(D^3(60-24D+87D^2-64D^3+159D^4-48D^5+113D^6+29D^8+8D^9)\right.$$
$$-\left(D^2(84+24D-369D^2+96D^3-1000D^4+144D^5-1410D^6\right.$$
$$+96D^7-884D^8+24D^9-213D^{10})$$
$$+4\left(D(3+171D^2+425D^4+565D^6+340D^8+80D^{10})\right.$$
$$\left.\left.\left.-3(1+D^2)^4(3+11D^2+3D^4)\tan^{-1}D\right)\tan^{-1}D\right)\tan^{-1}D\right)\Big]$$
$$+\frac{\alpha_1 D^4}{3E^2h^2\mu}\left[mD^2\left(5D+3D^3+3(1+D^2)^2\tan^{-1}D\right)\right.$$
$$-\frac{A_0 D}{6(1+D^2)}\left(D^2(1+D^2)(30+9D+67D^2+7D^3+31D^4)-(D(12\right.$$
$$+15D+15D^2-49D^3+67D^4-19D^5+13D^6-3D^7-3D^8)$$
$$\left.+6(1+D^2)^3(3-D+10D^2)\tan^{-1}D)\tan^{-1}D\right)$$
$$-\frac{{A_0}^2}{12m(1+D^2)^2}\left(18D^3+9D^4+33D^5+5D^6+19D^7-(54D^2-3D^3\right.$$
$$-81D^4+2D^5-8D^6+5D^7+19D^8$$
$$+6(1+D^2)^2\left(3D-D^2+24D^3+3(1+D^2)^2\tan^{-1}D\right)$$
$$\left.\left.\times\tan^{-1}D)\tan^{-1}D\right)\right],$$

(74)

$$F_3(D)=\left(1-\frac{3}{5}D^2\right)+\frac{3\alpha\sigma_0 D}{2}\left(\left(1-\frac{\alpha\sigma_0}{2}\right)-DQ_0\right)$$
$$-\frac{3(3\lambda_1+2\lambda_2)D}{2h^2\mu}\left[m-\frac{7Dm}{8}\left(\left(1-\frac{\alpha\sigma_0}{4}D\right)-\frac{2Q_0D}{3h}\right)\right.$$
$$\left.-\frac{Dm}{4}\left(\left(1-\frac{\alpha\sigma_0}{4}D\right)-\frac{2Q_0D}{3h}\right)^2\right]-\frac{3mD\alpha_1}{2h^2\mu}\left(1+D^2\right)$$

(75)

The above three weight functions are numerically evaluated for different values of $\alpha\sigma_0$ and the results are presented Figure 3.

5. Flow through a Channel with Smooth Constriction:

The above theory is applied to the problem of flow through a channel with smooth, axisymmetric constriction , defined in non-dimensional variables by [9]:

$$h(x)=H_0-\frac{\delta_m}{2}\left(1+\cos\frac{\pi x}{x_0}\right), \tag{76}$$

where δ_m is the maximum projection of the constriction and H_0 is the half width of the channel.

The third approximation method is valid, when the condition given in (57) is satisfied, i-e;

$$|D^n|=|h'(x)^n|<<1, \tag{77}$$

for some positive integer n. The expression for D, using equation (76), becomes:

$$D=h'(x)=\frac{\pi\delta_m}{2x_0}\sin\frac{\pi x}{x_0}. \tag{78}$$

We note that the condition $|D^n|<<1$ will be satisfied if the following non-dimensional quantities take the values:

$$L=4.0,$$
$$x_0=1.0,$$
$$\delta_m=0.32H_0,$$
$$H_0=1.0.$$

For these values $|D|$ has a maximum value of 0.5 at $x=x_0/2$ and satisfies the condition given in (59) for all positive values of n.

To determine the effect of porous layer parameter on flow in the channel near stenosis, it is important to determine the resistance to the flow and shear stress. The resistance to flow denoted by R.F is defined as:

$$R.F.=\frac{\text{averagepressuredropacrossthechannel}}{\text{fluxinthedirectionofflow}}. \tag{79}$$

To determine R.F. we need to firstly determine average pressure drop across the channel and flux in the direction of flow. The average pressure drop across the channel can be calculated form the following expression:

$$P_0-P=\frac{1}{2hL}\int_{-h(x)}^{h(x)}\int_0^L\frac{\partial p}{\partial x}\,dx\,dy. \tag{80}$$

Using equation (68) and integrating over the interval $-h$ to $+h$ equation (80) take the form:

$$p=\frac{3\mu m}{2hL}\int_0^L\frac{1}{h^2}\left(1-\frac{3D^2}{5}\right)dx+\frac{9\mu\alpha}{4hL}\int_0^L\frac{D\sigma_0}{h^2}\left[\left(1-\frac{\alpha\sigma_0}{4}D\right)-\frac{2}{3}QD\right]dx$$
$$+\frac{9m^2(3\alpha_1+2\alpha_2)D}{4hL}\int_0^L\frac{D}{h^4}dx$$
$$-\frac{9\alpha_1}{4hL}\int_0^L\frac{D}{h^4}\left[m^2-Dm\left(\left(1-\frac{\alpha\sigma_0}{4}D\right)-\frac{2}{3}QD\right)\right]dx.$$

(81)

The dimensionless momentum flux in the horizontal direction has the form:

$$M=\int_{-h}^{h}u^2\,dy, \tag{82}$$

using the expression for u from equation (66) in equation (82) and after performing the indicated integration, we find that:

$$M=\frac{\left(4\left(525+120D^2+64D^4\right)\mu^2+120AD\left(6D^2-35\right)\mu\alpha\sigma_0+9975A^2D^2\alpha\sigma_0^2\right)}{3500h}.$$

(83)

The expression for resistance to the flow, using equation (81) and equation (83) takes the following form :

$$R.F=\frac{P}{M} \tag{84}$$

where P and M are given by equations (81) and (83) respectively .

Analytical evaluation of integral given in numerator of equation (84) is complicated, therefore we have numerically evaluated it on a computer. The R.F was calculated for different values of $\alpha\sigma_0$, and is presented in Table 1 and figure 4.

Table 1: Resistance to the flow for different values of $\alpha\sigma_0$

$\alpha\sigma_0$	**h**	**D**	**R.F**
0.	0.84	0.5	1.01654
0.1	0.84	0.5	0.991569
0.2	0.84	0.5	0.941165
0.3	0.84	0.5	0.865944
0.4	0.84	0.5	0.769781
0.5	0.84	0.5	0.659201
0.6	0.84	0.5	0.541964
0.7	0.84	0.5	0.425457
0.8	0.84	0.5	0.315503
0.9	0.84	0.5	0.215862
1.0	0.84	0.5	0.128351

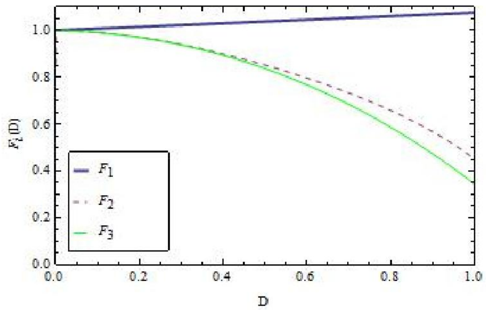

Figure 3: Comparison of weight functions for $\alpha\sigma_0 = 0.2$.

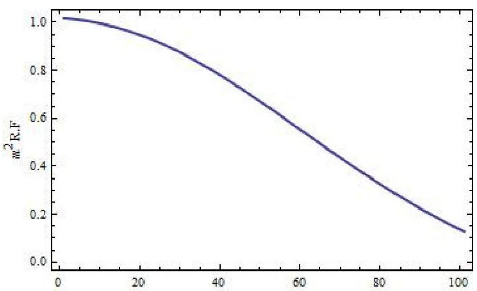

Figure 4: Resistance force vs permeability

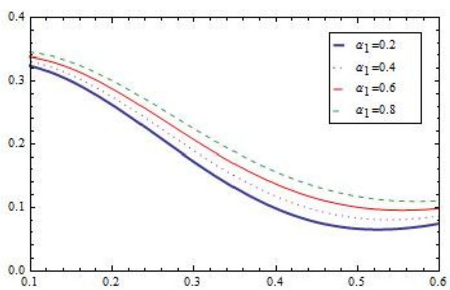

Figure 5: Variation of wall stress for different values of α_1.

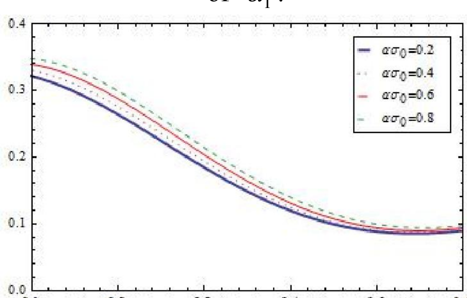

Figure 6: Variation of wall stress for different values of $\alpha\sigma_0$.

6 Results and Discussion

Three approximate methods based on different geometrical situations are used to solve two-dimensional second order nonlinear differential equations arising due to creeping flow of a Second Grade fluid through a channel of varying gap bounded below by a permeable bed. With the help of these methods we are able to have an idea about the geometrical effects of the curved channel on the flow which otherwise are not observed by using other methods. These methods provide an alternate approach to the conventional methods of solving a two dimensional problem. The weight functions F_1, F_2 and F_3 are calculated for each of the methods, which are graphically shown in figure 3. It is observed that F_2 and F_3 are influenced by $\alpha\sigma_0$ while F_1 is independent of $\alpha\sigma_0$. To study the effect of permeability on flow pattern in arteries having stenosis an idealized stenosed geometry is considered and resistance to the flow $(R.F)$ and shear stress are calculated. The obtained results are graphically evaluated for different values of $\alpha\sigma_0$ and non-Newtonian parameters α_1. The tabulated results for R.F are shown in table 1 which shows the effect of $\alpha\sigma_0$ on resistance to the flow and it is observed that as we increase the value of $\alpha\sigma_0$, R.F decreases as expected.

Figure 4 shows the behavior of R.F with $\alpha\sigma_0$, which clearly shows decrease in R.F due to increase in $\alpha\sigma_0$. In figures 5 and 6 the behavior of shear stress is shown for different values of α_1 and $\alpha\sigma_0$ respectivley. It is observed that as we increase the value of α_1 the shear stress increases. A similar effect is observed as the value of $\alpha\sigma_0$ is increased.

7. Conclusion

Creeping flow of an incompressible second grade fluid in a channel of varying width bounded below by a porous bed is solved using three approximate methods based on three different geometrical configurations. These methods provide an alternate approach to the conventional methods of solving two dimensional problem. With the help of these methods we are able to have an idea that the geometrical effects of curved channel on the flow can not be observed by using other methods. Expressions for velocity and pressure gradient are calculated for each method. Weight functions are calculated for all three methods and are evaluated for different values $\alpha\sigma_0$. It is observed F_2 and F_3 's functions are influenced by $\alpha\sigma_0$.

To study the effect of $\alpha\sigma_0$ on flow pattern in arteries having stenosis an idealized stenosis geometry is considered and resistance to the flow and shear stress are calculated. It is noticed that as we increase the value of $\alpha\sigma_0$ resistance to the flow in the stenosed region decreases as expected. However with increasing the values $\alpha\sigma_0$ and α_1 the shear stress increases in stenosed region. We can conclude that the over all effect of $\alpha\sigma_0$ is to decrease the resistance to the flow and thus to reduce the abnormalities due to irregular boundaries. It is also found that our results are more general as we can recover :

- solution obtained by Langlois as $\sigma \backslash \sigma_0 \to \infty$ and by setting $\alpha_1 = \alpha_2 = 0$.
- solution obtained by Rudraiah et al by setting $\alpha_1 = \alpha_2 = 0$.
- solution obtained by Siddiqui et al as $\sigma \backslash \sigma_0 \to \infty$.

We hope that this investigation may be helpful in evaluating the performance of various prosthetic devices that ultimately may be implemented into living system.

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