INTEGRATED SEMI GROUPS AND CAUCHY PROBLEM FOR SOME FRACTIONAL ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract: Let A be a linear closed operator defined on a dense set in a Banach space E to E. In this note it is supposed that A is the generator of $\alpha-$ times integrated semi group, where α is a positive number. The abstract Cauchy problem of the fractional differential equation: $\frac{d^\beta u(\tau)}{dt^\beta} == Au(t) + F(t)$, With the initial condition$u_0 \in E$, is studied, where $0 < \beta \leq 1$, and F is a given abstract function. An application is given.

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1. INTRODUCTION

The theory of integrated semi groups of operators on a Banach space were introduced by Arendt [1], [2]. Hieher [3] refined the theory by introducing $\alpha-$ times integrated semi groups for positive numbers.

Integrated semi groups are a natural extension of semi group theory to deal with operators that have polinomially bounded resolvent in a half plane. It is well known that the Schrodinger operator:

$$i\left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right]$$

Generates a co-semi group on $L^p(R^n)$ if and only if $p = 2$, (see Hormander [4,5 and 6]). But Hieber [3] showed that the Schrodinger operator generates an $\alpha-$ times integrated semi group on $L^p(R^n)$ for $\alpha > n\left|\frac{1}{n} - \frac{1}{p}\right|$, where R^n is the $n-$ dimensional Euclidean space and $L^p(R^n)$ is the set of all measurable functions f such that the integral $\int_{R^n}|f(x)|^p dx$ exists.

Denote by E a Banach space. Let $L(E) = L(E, E)$ be the space of bounded linear operators from E to E. Let $\{S(t), t \geq 0\}$ be a family of operators in $L(E)$. Suppose that A is a linear closed operator defined on a dense set $D(A)$ in E. The family $\{S(t), t \geq 0\}$ is called exponential bounded $\alpha-$ times integrated semi group generated by A if the following conditions are satisfied:

C_1: $\{S(t), t \geq 0\}$ is strongly continuous,

C_2: There exists $M > 0$ and a real number c such that

$\|S(t)\| \leq Me^{ct}, \quad t \geq 0,$

C_3: The interval (c, ∞) is contained in the resolvent set $\rho(A)$ of A and

C_4:

$$(\lambda I - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t) dt,$$

For all $\lambda > c$, where I is the identity operator (see [7], [8],and [9]).

Under the conditions $C_1, \ldots, C_4$ we shall solve in section 2 the following Caushy problem:

$$\frac{d^\beta u(t)}{dt^\beta} = Au(t), \qquad t > 0, \tag{1.1}$$

$$u(0) = u_0 \in D(A), \tag{1.2}$$

Where $0 < \beta \leq 1$.

Recall the definition of fractional derivatives, one of the definitions of the fractional derivative $\frac{d^\beta}{dt^\beta}$ is given by

$$\frac{d^\beta f(t)}{dt^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^n(s)}{(t-s)^{n-1+\beta}} ds + \sum_{k=0}^{n} f^k(0^+)\Phi_{k-\beta+1}(t),$$

Where $n - 1 \leq \beta < n, \Phi_c(t) = \frac{t_+^c}{\Gamma(c)}$, $t_+^c = t^c H(t), H(t)$ being the Heaviside function and $\Gamma(c)$ is the gamma function (see [5], [6]).

2. REPRESENTATION OF THE SOLUTION

Let us solve the Cauchy problem (1.1), (1.2) under the conditions $C_1, \ldots, C_4$. It is suitable to rewrite the Cauchy problem (1.1), (1.2) in the form:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t \frac{Au(s)}{(t-s)^{1-\beta}} ds. \tag{2.1}$$

By a solution (2.1), we mean a function u such that:

1. u is continuous on $J=[0,T]$, u is an element of $D(A)$, for each t in J and $Au(t)$ is continuous on J.

2. u satisfies equation (2.1).

Theorem 2.1. If the conditions $C_1,\dots,C_4$ are satisfied and u_0 is a given element in $D(A)$, then the unique solution of (2.1) is represented by:

$$u(t)=\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty \xi_\beta(s)S(t^\beta s)u_0ds, \qquad (2.2)$$

Where $\xi_\beta(s)$ is a probability density function defined on $(0,\infty)$, $0<\beta\le 1$ and $n-1<\alpha\le n$.

Proof. Applying formally the Laplace transform

$$v(p)=\int_0^\infty e^{-Pt}u(t)dt, \qquad P>0$$

To (2.1) yields

$$v(P)=p^{\beta+\alpha\beta-1}\int_0^\infty e^{-P^\beta t}S(t)u_0dt. \qquad (2.3)$$

Consider the one-sided stable probability density function $\rho_\beta(t)$, whose Laplace transform is given by

$$\int_0^\infty \rho_\beta(t)e^{-Pt}dt=e^{-P^\beta},$$

Consequently

$$\int_0^\infty \rho_\beta(t)e^{-P\theta^{\frac{1}{\beta}}}dt=e^{-p^\beta\theta}\ . \qquad (2.4)$$

Differentiating both sides of (2.4) with respect to p, we get

$$\int_0^\infty t\rho_\beta(t)e^{-P\theta^{\frac{1}{\beta}}t}dt=\beta\theta^{1-\frac{1}{\beta}}p^{\beta-1}e^{-P^\beta\theta} \qquad (2.5)$$

From (2.3) and (2.5), one gets:

$$v(p)=p\int_0^\infty e^{-Pt}\left[\int_0^\infty \xi_\beta(\theta)S(t^\beta\theta)u_0d\theta\right]dt, \quad (2.6)$$

Where

$$\xi_\beta(t)=\frac{1}{\beta}t^{-1-\frac{1}{\beta}}\rho_\beta\left(t^{-\frac{1}{\beta}}\right).$$

Notice that $\xi_\alpha(t)$ is a probability density function defined on $[0,\infty]$. The Laplace transform of ξ_β is given by

$$\int_0^\infty e^{-Pt}\xi_\beta(t)dt=\sum_{j=0}^\infty\frac{(-p)^j}{\Gamma(1+j\beta)}.$$

We have

$$S(t)u_0=\frac{t^\alpha}{\Gamma(\alpha+1)}u_0+\int_0^t S(s)Au_0ds, \qquad (2.7)$$

For all $t>0$ and $u_0\in D(A)$,

Since $u_0\in D(A^n)$, $n-1<\alpha\le n$, one gets from (2.7)

$$\frac{d^kS(t)}{dt^k}u_0=0,\ \text{at}\ t=0,\ \ k=0,1,\dots,n-1 \qquad (2.8)$$

Remembering the simple fact about the Laplace transform of the fractional derivatives and using (2.6), (2.8), one get

$$u(t)=\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty \xi_\beta(s)S(t^\beta s)u_0ds$$

Hence the required result.

Noticing that:

$$\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}t^{\alpha\beta}=\Gamma(\alpha\beta+1), \qquad \int_0^\infty \theta^\alpha\xi_\beta(\theta)d\theta=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\beta+1)}$$

And using (2.2), (2.7), we get

$$u(t)=u_0+\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty\int_0^{t^\beta\theta}\xi_\beta S(s)Au_0dsd\theta. \qquad (2.9)$$

3. NON HOMOGENEOUS EQUATIONS

Let us consider the nonhomogeneous equation

$$\frac{d^\beta u(t)}{dt^\beta}=Au(t)+f(t), \qquad (3.1)$$

With the initial condition

$$u(0)=u_0\in D(A^n), \qquad (3.2)$$

Where f is a given abstract function defined on J and with values in E.

Theorem 3.1.

If the conditions $C_1,\dots,C_4$ are satisfied, $u_0\in D(A^n)$, $n=1,2,\dots\dots$ and $f(t)\in D(A^n)$ for every $t\in J$, $n-1<\alpha\le n$, then the solution of the Caushy problem (3.1), (3.2) is given by:

$$u(t)=u^*(t)+F(t),$$

Where $u^*(t)$ is given by formula (2.2) or (2.9) and

$$F(t)=\beta\int_0^t\int_0^\infty\frac{d^{\alpha\beta}}{d\theta^{\alpha\beta}}\theta\xi_\beta(\theta)\eta^{\beta-1}S(\eta^\beta\theta)f(t-\eta)d\theta d\eta.$$

Proof. If v and g are the Laplace transform of u and f, respectively, then

$$v(p)=p^{\beta-1}\left(p^\beta I-A\right)^{-1}u_0+\left(p^\beta I-A\right)^{-1}g(p),$$

So

$$v(p)=p^{\alpha\beta}p^{\beta-1}\int_0^\infty e^{-\rho^\beta t}S(t)g(p)xdt$$
$$+p^{\alpha\beta}\int_0^\infty e^{-p^\beta t}S(t)g(p)dt.$$

Using techniques similar to the techniques which are used in theorem (1.1), we get

$$L^{-1}\left[p\int_0^\infty e^{-p^\beta t}S(t)dt\right]=\frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\int_0^\infty \theta\xi_\beta(\theta)S(t^\beta\theta)xd\theta,$$

For every element $x\in D(A^{n+1})$, where L^{-1} is the inverse Laplace transform of L.

Thus

$$F(t)=\beta\int_0^t\int_0^\infty \frac{d^{\alpha\beta}}{dt^{\alpha\beta}}\theta\xi_\beta(\theta)\eta^{\beta-1}S(\eta^\beta\theta)f(t-\eta)\,d\theta d\eta.$$

Hence required result, see [10-16].

4. APPLICATION

Let $p>1, 0<\alpha\le\frac{p-1}{p}, E=L^p[0,1],$

Define the operator A by

$$(Ag)(x)=-\frac{dg(x)}{dx}+\frac{\alpha}{x}g(x),$$

Where $D(A)$ is the set of all absolutely continuous functions g defined on the interval $[0,1]$ with $g(0)=0$ and $\frac{dg(x)}{dx}\in L^P[0,1]$.

The considered operator A generates the integrated semi group $S(t)$, where

$$[S(t)g](x)=\int_0^t x^\alpha(x-s)^{-\alpha}g(x-s)H(x-s)ds\,,\quad(4.1)$$

$x\in[0,1]$, H is the Heaviside function, see [7], notice that $S(t)$ is not a semi group.

Consider now the following Cauchy problem

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta}=-\frac{\partial u(x,t)}{\partial x}+\frac{\alpha}{x}u(x,t),\qquad(4.2)$$

$$u(x,0)=u_0(x),\qquad(4.3)$$

Where $u_0(x)\in D(A)$.

Using formula $(2.2),(3.1)$, we can solve the Cauchy problem$(4.2),(4.3)$ in $D(A)$.

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