A Class of Exact Solutions of the Navier-Stokes Equation in Porous Medium

Murad Ullah[1*], S. Islam[2], M. K. Alam[3], Tariq Rahim[3], Muneeb Ur Rehman[1], A. M. Siddiqui[4]

[1]Department of Mathematics, Islamia College (Charted University), Peshawar, KPK, Pakistan.
[2]Department of Mathematics, Abdul Wali Khan University Mardan, KPK, Pakistan.
[3]Department of Mathematics,National University of Computer and Emerging Sciences, Peshawar, Pakistan.
[4]Department of Mathematics Pennsylvania State University, York Campus,
1031 Edgecombe Avenue, New York, PA 17403, USA
*Telephone# +92-313-9911358, Email: muradullah90@yahoo.com
muradullah90@yahoo.com

Abstract: The aim of the present study is to find the exact solutions of three dimensional incompressible, unsteady viscous fluids in porous medium. This flow field represents several shear layers imposed on an irrotational, three dimensional straining flow. A class of exact solutions of the Navier Stokes equations is obtained for a general initial condition through porous medium.
[Murad Ullah, S. Islam, M. K. Alam, Tariq Rahim, Muneeb Ur Rehman, A. M. Siddiqu. **A Class of Exact Solutions of the Navier-Stokes Equation in Porous Medium.** *Life Sci J* 2013;10(1s):104-109] (ISSN:1097-8135). http://www.lifesciencesite.com. 16

Keywords: Exact solutions, Vorticity, porosity, Navier-Stokes equations.

1. Introduction

The fundamental governing equations for fluid motions are the Navier-Stokes equations. These equations are non-linear and only a small number of exact solutions have been found. Current advances in computer technology make the complete numerical integration of the Navier Stokes equations more feasible but the accuracy of the results can only be ascertained by a comparison with an exact solution. First, these solutions represent fundamental fluid dynamic flows and second, these solutions serve as standards for checking the accuracies of the many approximate methods, whether they are numerical, asymptotic or empirical. Due to this reason, the inverse solutions for the Newtonian fluids have become attractive because of its procedure for solving different type of problems. The procedure not only simplifies the partial differentials equation but also guide towards the exact solutions.

Porous media processes are encountered in numerous real life examples such as sub-surface flow, reactive flow and bioremediations in soils, as well as medical applications. One particular example is encountered in the petroleum engineering, where oil (the nonwetting phase) inside a reservoir is displaced by water in the process of oil production.

Employing the modified Darcy's Law [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], the two-dimensional unsteady flow of an incompressible homogenous fluid flow in a porous medium in the absence of body force, is governed by the equations, written in Cartesian coordinates (x, y),

$$\nabla \cdot \tilde{u} = 0, \quad \text{(continuity)} \qquad (1)$$

$$\rho\left(\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla \tilde{u})\right) = -\nabla p + \mu \nabla^2 \tilde{u} + r, \qquad (2)$$

(momentum)

where $\tilde{u}$ is the velocity vector, p the pressure, μ the viscosity, and r is the Darcy's resistance given by the relation

$$r = -\frac{\bar{\mu}\tilde{u}_d}{K}. \qquad \text{(2-A)}$$

Here K is the permeability and $\bar{\mu}$ the effective viscosity of the fluid in the porous medium, and $\tilde{u}_d$ is the Darcian velocity which is related to the fluid velocity $\tilde{u}$ by $\tilde{u}_d = \tilde{u}\phi$, $(0 < \phi < 1)$, ϕ being the porosity of the medium. In general, the effective viscosity $\bar{\mu}$ and the fluid viscosity μ are different. However, at the macro level we may take them equal, through this assumption does not hold at the micro level. Following [4, 12, 13], equation (2-A) can now be written as

$$r = -\frac{\mu\phi\tilde{u}_d}{K}. \qquad \text{(2-B)}$$

Stretching, convection, viscous diffusion and porosity of vorticity are the elemental processes for the vortex motions of a viscous fluid in a free space. By introducing the vorticity $\boldsymbol{\omega}$, the Navier Stokes equations for an incompressible fluid and the continuity equation for the velocity $\boldsymbol{v}$ are reduced to the following equation:

$$\frac{\partial \omega}{\partial t} - (\boldsymbol{\omega}.\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{\omega} + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega} - \frac{\nu}{K}\boldsymbol{\omega} \qquad (3)$$

$$div\, v = 0 \qquad (4)$$

Remark 1: for $K \to \infty$, we get the results of (Kambe 1986) for incompressible Navier-Stocks equation.

where $\omega = rot\, v$, ν is the kinematic viscosity and K is the permeability constant. The four terms on the right hand side of equation (3) mentioned above shows the four processes. Batchelor [14] studied the steady state solutions for such vertex motions. Two dimensional unsteady problems was investigated by Kambe [17] and provided exact solutions for n general initial condition. The present work is the extension and generalization of the previous work of Kambe [20] with porous media. Townsend [21] considered $\frac{\partial}{\partial t} = 0$ and investigated the steady state solution for incompressible viscose fluid. Batchelor [14] also discussed the same solution for steady state Navier-Stock equations. The motion of the small scale shear layers convected by a flow of large scale might be considered as a local model of turbulence. Rose and Sulem [22] considered that non-local interaction between larger and smaller eddies in wave number space is important for enstrophy transfer in the two dimensional turbulence. Kambe and Takao [16] found the experimental evidence of interactions and collision of vertices and showed that the two vertex rings moving in parallel come into contact in the course of time and result in information of a single dumbbell-shaped vortex loop. Viscous effect would be important at the point of contact of two antiparallel vortex lines. The study of acoustic emission by the vortex collision was the main objective of Kambe and Minota [17].

The present paper is the extension of Kambe's paper [20], where he obtained a class of exact solutions of the Navier Stokes equations. We study the similar problem under the influence of porous medium for three dimensional incompressible fluid flows. Formulation of the problem is given in section 2 and section 3 contains different solutions to the problem. Conclusion is given in section 4.

2. Formulation of an initial value problem

For an incompressible viscous flow the components of velocity are defined in three-dimensional Cartesian coordinate system *(x, y, z)* as

Consider the velocity components as

$$u = -a(t), \qquad v = b(t)y + V(x,t), \qquad w = c(t)z + W(x,t), \qquad (5)$$

Where *a, b, c* are functions of time *t* and *v(x, t)* and *w(x, t)* are unknown functions. It is assumed that the three parameter functions always satisfy the relation

$$a(t) = b(t) + c(t). \qquad (6)$$

The equation of continuity (4) is satisfied by putting equation (5) and then using equation (6). The vorticity derived from the velocity (5) has only *y, z* components, which are independent of *y, z* which is as follows

$$\omega = rot\, v = (\,0,\ \ \omega_y(x,t),\ \ \omega_z(x,t)\,) \qquad (7)$$

where

$$\omega_y = -\frac{\partial W}{\partial x}, \qquad \omega_z = \frac{\partial V}{\partial x},$$

The vorticity equal to zero when both *V* and *W* vanish and the equation (5) reduces to the irrotational field

$$v_s = (-ax, by, cz). \qquad (9)$$

This represents a stagnation point flow at the origin, which inflows both positive and negative x-directions and outflows to the y and z-directions.

Substitution of v and ω, by putting equation (5) and (7) in (3), we get the following components equations

$$\frac{\partial \omega_y}{\partial t} - \omega_y\left(b - \frac{\nu}{K}\right) - ax\frac{\partial \omega_y}{\partial x} = \nu\frac{\partial^2 \omega_y}{\partial x^2} \qquad (10)$$

$$\frac{\partial \omega_z}{\partial t} - \omega_z\left(c - \frac{\nu}{K}\right) - ax\frac{\partial \omega_z}{\partial x} = \nu\frac{\partial^2 \omega_z}{\partial x^2}$$

Remark 2: In equations (10) and (11) by putting $K \to \infty$, we get the results of (Kambe 1986) for incompressible Navier-Stocks equation.

On the left hand side of both the equations the second and third term represents stretching and convection of vortex lines respectively, whereas the right hand side represents the viscous diffusion. The equation of ω_y and ω_z are separated and independently solved as far as relation (4) is satisfied. Let us introduce the functions $A_1(t), B_1(t)$ and $C_1(t)$ by

$$A_1(t) = \int_0^t \left(a(s) - \frac{\nu}{K}\right)ds, \quad B_1(t) = \int_0^t \left(b(s) - \frac{\nu}{K}\right)ds, \quad C_1(t) = \int_0^t \left(c(s) - \frac{\nu}{K}\right)ds, \qquad (12)$$

Remark 3: by putting $K \to \infty$ in equation (12), we get the results of (Kambe 1986) for incompressible Navier-Stocks equation.

Relation (6) leads to the equation $A_1 = B_1 + C_1 + \frac{\nu}{K}t$

The first two terms of the equations (10) can be written as

$$\frac{\partial \omega_y}{\partial t} - \omega_y\left(b - \frac{\nu}{K}\right) = e^{B_1(t)}[\,e^{-B_1(t)}\omega_y] \qquad (13)$$

Similarly the first two terms of equation (11) can be written as

$$\frac{\partial \omega_z}{\partial t} - \omega_z\left(c - \frac{\nu}{K}\right) = e^{C_1(t)}[\,e^{-B_1(t)}\omega_y] \qquad (14)$$

Remark 4: Equations (13) and (14) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$.

Let by defining the following transformation

$$f = e^{-B_1(t)}\omega_y, \qquad g = e^{-B_1(t)}\omega_y, \qquad (15)$$

$\xi = e^{A_1(t)}x,$ $\tau = \int_0^t e^{A_1(t)}ds$ (16)

By applying the above transformation, equations (13) and (14) takes the form of diffusion equation:

$$\frac{\partial f}{\partial \tau} = \nu \frac{\partial^2 f}{\partial \xi^2}, \quad (17)$$

$$\frac{\partial g}{\partial \tau} = \nu \frac{\partial^2 g}{\partial \xi^2}, \quad (18)$$

The function f and g in equation (15) satisfying the above diffusion equations and the following initial conditions:

$f\,(at\,\tau = 0) = f_0(\xi)$, or $\omega_y(x,0) = f_0(x)$, (19)

$g\,(at\,\tau = 0) = g_0(\xi)$, or $\omega_z(x,0) = g_0(x)$, (20)

3. Solution Methodology

We now solve the diffusion equation $\frac{\partial f}{\partial \tau} = \nu \frac{\partial^2 f}{\partial \xi^2}$ subject to the initial conditions $f\,(at\,\tau = 0) = f_0(\xi)$, or $\omega_y(x,0) = f_0(x)$.

For this we consider the function $f = \frac{1}{\sqrt{\tau}}\exp\left(\frac{-\xi^2}{4\nu\tau}\right)$, which is the fundamental solution of the diffusion equation (17), known as the heat kernel.

Equation (17) and (19) satisfying by the solution has the form

$$f(\xi,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} f_0(\acute{\xi})\exp\left[\frac{-(\acute{\xi}-\xi)^2}{4\nu\tau}\right]d\acute{\xi}$$

Since $\omega_y = f e^{B_1(t)}$, so the expression for vorticity ω_y has the form:

$$\omega_y = \frac{1}{\sqrt{4\nu\pi\tau}}e^{B_1(t)}\int_{-\infty}^{\infty} f_0(\acute{\xi})\exp\left[\frac{-(\acute{\xi}-\xi)^2}{4\nu\tau}\right]d\acute{\xi} \quad (21)$$

Similarly equation (18) subject to the initial condition (20) can be solved and expressed as

$$\omega_z = \frac{1}{\sqrt{4\nu\pi\tau}}e^{C_1(t)}\int_{-\infty}^{\infty} g_0(\acute{\xi})\exp\left[\frac{-(\acute{\xi}-\xi)^2}{4\nu\tau}\right]d\acute{\xi} \quad (22)$$

Remark 5: Equations (21) and (22) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in ξ, $B_1(t)$ and in $C_1(t)$.

Equations (21) and (22) represent the exact solutions of initial value problem (10), (11), (19) and (20) of the Navier Stokes Equation.

3.1. Solutions to velocities V and W

We have

$$\omega_y = -\frac{\partial W}{\partial x}, \qquad \omega_z = \frac{\partial V}{\partial x},$$

The velocities V and W of the shear flow are given by $V(x,t) = \int_{c_1}^{x}\frac{\partial W}{\partial x}(\acute{x},t)d\acute{x}$ which leads to the vorticity form

$$V(x,t) = \int_{c_1}^{x}\omega_z(\acute{x},t)d\acute{x} \quad (23)$$

$W(x,t) = \int_{c_1}^{x}\frac{\partial V}{\partial x}(\acute{x},t)d\acute{x}$ which leads to the vorticity form

$$W(x,t) = -\int_{c_1}^{x}\omega_y(\acute{x},t)d\acute{x} \quad (24)$$

Where the constant c_1 and c_2 are chosen to satisfy the boundary conditions. When a, b and c are all constants, we have

(i) We know that $A_1(t) = \int_0^t\left(a(s) - \frac{\nu}{K}\right)ds,$ where a is a constant, then

$$A_1(t) = \left(a - \frac{\nu}{K}\right)t \quad (25)$$

(ii) We know that $B_1(t) = \int_0^t\left(b(s) - \frac{\nu}{K}\right)ds,$ where a is a constant, then

$$B_1(t) = \left(b - \frac{\nu}{K}\right)t \quad (26)$$

(iii) We know that $C_1(t) = \int_0^t\left(c(s) - \frac{\nu}{K}\right)ds,$ where a is a constant, then

$$C_1(t) = \left(c - \frac{\nu}{K}\right)t \quad (27)$$

(iv) $\xi = e^{\left(a-\frac{\nu}{K}\right)t}x,$ and $\tau = \frac{1}{2\left(a-\frac{\nu}{K}\right)}\left(e^{2\left(a-\frac{\nu}{K}\right)t} - 1\right)$ (28)

Remark 6: Equations (25)-(28) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting the permeability constant $K \to \infty$.

Assuming deformation of a material volume ΔV, the flow general property is examined which is parallelepiped at an initial instant. Suppose that the parallelepiped at t=0 is enclosed by the two planes $x = x_1(0)$ and $x_2(0)$, $y = y_1(0)$ and $y_2(0)$ and $z = z_1(0)$ and $z_2(0)$ in the x, y and z-direction respectively.

Hence the length of the edges is

$(\Delta x)_0 = x_2(0) - x_1(0),$
$(\Delta y)_0 = y_2(0) - y_1(0),$
$(\Delta z)_0 = z_2(0) - z_1(0),$

respectively, which are all assumed positive. These planes move with the material particle and take the positions $x_i(t)$, $y_i(t,x)$ and $z_i(t,x)$ where $(i = 1,2)$ at the same time t. Since the equation of motion of the planes are given by

$$u = -a(t), \qquad v = b(t)y + V(x,t), \qquad w = c(t)z + W(x,t),$$

$$\frac{dx_i}{dt} = -a(t)x_i, \quad \frac{dy_i}{dt} = b(t)y_i + V(x,t), \quad \frac{dz_i}{dt} = c(t)z_i + W(x,t) \; (i = 1,2)$$

The rate of change of the distance for each pair of planes, i.e.

$\Delta x = x_2 - x_1,$ $\Delta y = y_2 - y_1,$ $\Delta z = z_2 - z_1,$ is governed by $\frac{d\Delta x}{dt} = -a\Delta x,$ $\frac{d}{dt}\Delta y = b\Delta y,$ $\frac{d}{dt}\Delta z = c\Delta z,$ respectively. By integrating these equation we get

$\Delta x = (\Delta x)_0 e^{-A(t)-\frac{\nu}{K}t}$, $\Delta y = (\Delta y)_0 e^{B(t)+\frac{\nu}{K}t}$, $\Delta z = (\Delta z)_0 e^{C(t)+\frac{\nu}{K}t}$ (29)

Remark 7: Equations (29) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$.

This represents contraction in the x-direction and stretching in the z-direction; further the relation $A_1 = B_1 + C_1 + \frac{\nu}{K}t$ by (4) leads to the equation

$\Delta x \Delta y \Delta z = (\Delta x)_0 (\Delta y)_0 (\Delta z)_0$ (30)

Along each axis the distances $\Delta x,\ \Delta y\ and\ \Delta z$ are uniform; by deforming the volume ΔV by the x dependent shear flow $(0, V, W)$, the material volume is conserved and $\Delta x,\ \Delta y\ and\ \Delta z$ may take any arbitrary finite values and ΔV are independent of its initial position.

3.2. Parallel shear layers

In this section we consider a particular case of g_0 and $f_0 = 0$ which is given by the delta functions:

$f_0 = 0, \quad g_0 = \sum_{i=1}^{N} V_i \delta(x - l_i)$ (31)

At $x = l_i (i = 1, 2, \dots, N)$ this represents N vertex sheets and the velocity distribution is given by

$V(x, 0) = \sum_{i=1}^{N} V_i H(x - l_i) + V_s$ (32)

where $c_1 \sum_{i=1}^{N} V_i = V_s$, V_s is constant and H is the unit step function:

$$H(x) = 0 (x < 0)$$
$$= 1 (x > 0)$$

and $W(x, 0) = 0$

The velocity of the flow is uniform except at $x = l_i$ where the velocity changes discontinuously and directed in the y-axis. This means the plane $x = l_i$ is a vortex sheet which is characterized by the strength V_i per unit y-length (V_i has the dimension of velocity) and by the vortex direction in the z-axis. Now we find the corresponding solution by introducing the initial condition (28) into equation (22).

$\omega_z(\xi, \tau) = \frac{1}{\sqrt{\pi}\lambda} e^{C_1(t)} \sum_{i=1}^{N} V_i \exp\left[\frac{-(l_i - \xi)^2}{\lambda^2}\right]$ and (33)

$\omega_y = 0$

By using the original variables we find the expression for ω_z in terms of x, instead of ξ.

Substituting $d = \lambda e^{-A_1}$, $L_i = l_i e^{-A_1}$ and $\xi = e^{A_1} x$ in equation (33)

$\omega_z(\xi, \tau) = \frac{1}{\sqrt{\pi} d} e^{C_1 - A_1} \sum_{i=1}^{N} V_i \exp\left[\frac{-(x - L_i)^2}{d^2}\right]$

(34)

Remark 8: Equations (33) and (34) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in $C_1(t)$, A_1 and in ξ.

From the second equation of (34) i.e. $L_i = l_i e^{-A_1}$ we get

$\frac{d}{dt} L_i = -(\alpha - \frac{\nu}{K}) L_i$

This means that the peak position $x = L_i$ of each term in the summation of (34) is convected toward $x = 0$ by the flow v_s. In addition to this convection effect, the solution (34) includes the effects of vortex stretching and viscous diffusion.

3.3. Single shear layer ($N = 1$)

By putting $N = 1, l_1 = l$ and $V_1 = V_0$ in (31), yielding $g_0 = V_0 \delta(x - l)$.

The corresponding velocity distribution at $t = 0$ is

$V(x, 0) - V_0 (H(x - l) - \frac{1}{2})$

with $V_s = \frac{1}{2} V_0$ in (32). At $x = l$, this represents vertex sheet of strength V_0 and the velocity takes the values $-\frac{1}{2} V_0$ and $\frac{1}{2} V_0$ on two sides. The initial condition solution is given by

$\omega_z(x, \tau) = \frac{V_0}{\sqrt{\pi} d} e^{C_1 - A_1} \exp\left[\frac{-(x - L)^2}{d^2}\right]$ (35)

From (35), where $L = l_i e^{-A_1}$, substituting this into (23) with $L = c_1$, we obtain the velocity

$V(x, t) = \frac{1}{2} V_0 e^{C_1 - A_1} erf\left(\frac{(x - L)}{d}\right)$

Where the introducing function is

$\frac{d}{d\acute{x}} erf(\alpha \acute{x}) = \frac{2\alpha}{\sqrt{\pi}} exp(-\alpha^2 (\acute{x})^2)$, where $\alpha = \frac{1}{d}$ and $\acute{x} = x - L$.

Remark 9: Equations (35)* and the velocity $V(x, t)$ reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in $C_1(t)$ and in A_1.

3.4. Eulerian strength Γ.

Global feature of the flow field is considered to be examined by assuming change in the strength of the shear layer. A fixed zonal region D is taken up in the x, y plane, which is defined by a unit width in the y -direction and infinite length in the x -direction $(-\infty < x < +\infty)$. The (Eulerian) strength Γ of the shear layer included in D is given by

$$\Gamma = \iint_D \omega_z dx dy \quad (36)$$

$$- \int_{y=0}^{1} \int_{x=-\infty}^{\infty} \omega_z dx dy \quad (37)$$

$$\Gamma = \int_{y=0}^{1} \int_{x=-\infty}^{\infty} \frac{\partial V}{\partial x} dx dy \quad (38)$$

Since

$V(x, t) = \frac{1}{2} V_0 e^{C_1 - A_1} erf\left(\frac{(x - L)}{d}\right)$ (39)

where $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) ds$.

The above equation can also be written as

$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-s^2) ds$ (40)

or

$$er\,f(x) = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$

Hence

$$V(x,t) = \frac{1}{2}V_0 e^{C_1 - A_1} \qquad (41)$$

Substituting (41) in (38) we get

$$\Gamma = V_0 e^{C_1 - A_1} \qquad (42)$$

Remark 10: Equations (42) reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in $C_1(t)$ and in A_1.

Since the vortex behavior is clearly understood in its Lagrangian aspect, we consider the strength of the shear layer which is estimated in a material area moving with the fluid particles. Let the material area which occupied the fixed region D at $t = 0$ be denoted by D_*. According to the second equation of (29), the width Δy of D_* in the y-direction grows by the factor e^B, which is independent of x.

3.5. Strength Γ_*

Since Δy is equal to e^B and ω_z does not depend upon y, in D_* the strength Γ_* of the shear layer becomes e^{B_1} times Γ.

$$\Gamma_* = e^{B_1}$$

$$\Gamma_* = e^{A_1 - C_1 - \frac{\nu}{K}t}\Gamma \quad \text{since} \quad A_1 = B_1 + C_1 + \frac{\nu}{K}t$$

(43)

We have

$$\Gamma_* = e^{A_1 - C_1 - \frac{\nu}{K}t} V_0 e^{C_1 - A_1}$$

Remark 11: The above equations reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in $C_1(t)$ and in A_1.

Since the assumption was at $t = 0$, so

$$\Gamma_* = V_0\,(const)$$

Thus we obtain Γ_*, the strength of the shear layer in D_* is invariant. This property does not depend on the z-position of D since ω_z is independent of z.

3.6. Enstrophy

The enstrophy is define by

$$Q = \iint_{D_*} \frac{1}{2}\omega^2 dxdy$$

Using (33)* and $\omega^2 = \omega_z{}^2$, we obtained Q with suffix 1,

$$Q_1 = \frac{V_0^2}{2\sqrt{2\pi}\lambda} e^{C(t) - \frac{\nu}{K}t}$$

The above equation shows that, since $\lambda = \sqrt{4\nu\tau(t)}$, always in the case $c = 0$ the enstrophy Q_1 decrease. This is only occurs when in the two dimensional flow vortex stretching is absent and provided that a = constant, Q_1 decays like $e^{-c(t)}$ as $t \to \infty$.

Next consider a particular case of $b = 0$ and $a = c = constant$. This corresponds to the case that $B_1 = 0$ and $A_1 = C_1 + \frac{\nu}{K}t$. Equation (35)* reduced to

$$\omega_z(x,t) = \frac{V_0}{\sqrt{\pi}d} e^{-\frac{\nu}{K}t} \exp\left[\frac{-(x-L)^2}{d^2}\right] \qquad (44)$$

Also $V(x,t)$ reduced to

$$V(x,t) = \frac{1}{2}V_0 e^{-\frac{\nu}{K}t} er\,f\left(\frac{(x-L)}{d}\right)$$

This solution tends to a steady state solution. Since $d = \lambda e^{-A_1}$, by first equation of (34), so

$$\sqrt{4\nu\tau e^{-2at}} = \sqrt{\left(\frac{2\nu}{a - \frac{\nu}{K}}\right)\left(1 - e^{-2\left(a - \frac{\nu}{K}\right)t}\right)}$$

From the first equation of (34) and (28) we have

$$d_\infty = \lim_{t \to \infty} d = \sqrt{\left(\frac{2\nu}{a - \frac{\nu}{K}}\right)}$$

Using this and the property $L \to 0$ and $t \to \infty$, equation (44) tends to

$$\omega_z(x,t) = \frac{V_c}{\sqrt{\pi}d_\infty} \exp\left[\frac{-(x)^2}{d_\infty{}^2}\right] \equiv \omega_0(x)$$

It can be shown that the last form $\omega_0(x)$ is obtained from (11) with $\frac{\partial}{\partial t} = 0$ and $a = c$,

i.e. $\frac{\partial(axw_z)}{\partial x} + \nu\frac{\partial^2 \omega_z}{\partial x^2} = 0$

This is readily integrated once with integration constant F and the resulting first order equation vorticity diffusion, while the vortex stretching and the inward convection are represented by the parameter a. The parameter $\frac{\nu}{K}$ in d_∞ represents the magnitude of the porosity. Their balance determines d_∞.

Remark 12: The above result reduced to the results of (Kambe 1986) for incompressible Navier-Stocks equation by putting $K \to \infty$ in each and every equation where K is involve.

4. Conclusions

The present work in this paper shows the exact solutions for an unsteady motion of shear layer convected by a three dimensional irrotational straining flow. The four aspects in the solution are discussed which are vortex motion i.e vortex stretching, viscous diffusion, convection and porosity of vorticity. The solution in the present paper is the three-dimensional generalization with porosity of the (Kambe 1986) for incompressible Navier-Stocks equation.

Throughout in this paper when we put the permeability constant $K \to \infty$, we get the solutions of (Kambe 1986) for incompressible Navier-Stocks equation.

References

1. Khuzhayorov, B., Auriault, J. L. and Royer, P. (2000). Derivation of Macroscopic Filtration Law for Transient Linear Viscoelastic Fluid Flow in Porous Media, Int. J. Engng Sci., vol. 38, 487-504.
2. Kamel, M. T. and Hamdan, M. H., 2006. Riabounchisky Flow through Porous Media, Int. J. of Pure and Appl. Math. vol. 27(1), pp. 113-126.
3. Merabet, N., Siyyam, H., and Hamdan, M.H., 2008. Analytical Approach to the Darcy-Lapwood-Brinkman Equation, Appl. Math. and Comput., vol. 196(2-1), pp. 679-685.
4. Vafai, K. and Tien, C.L., 1981. Boundary and Inertia Effects on Flow and Heat Transfer in Porous Media, Int. J. Heat Mass Trans., vol. 24, pp. 211-220.
5. Vafai, K. and Kim, S.J., 1995. On the Limitations of Brinkman–Forchheimer-Extended Darcy Equation, Int. J. Heat Fluid Flow, vol. 16, pp. 11-15.
6. Vafai, K., 2005. Hand book of Porous Media, IInd edition, CRC press, Taylor & Francis Group, Boca Raton, FL 33487-2742, LLC.
7. Kaviany, M., 1985. Laminar Flow Through a Porous Channel Bounded by Isothermal Parallel Plates, Int. J. Heat Mass Trans., vol. 28, pp. 851.
8. Siddiqui, A.M., Islam, S. and Ghori, Q. K., 2006. Two-Dimensional Viscous Incompressible flows in a Porous Medium, J. of Porous Media, vol. 9(6), pp. 591-596.
9. Islam, S. and Zhou, C.Y., 2006. Exact Solutions of a Second Grade Fluid in a Porous Medium, Proc. NSC, Beijing.
10. Islam, S. and Zhou, C.Y., 2007. Certain inverse solutions of a second grade MHD aligned fluid flows in a porous medium, Journal of Porous Media, vol. 10(4), 401-408.
11. Islam, S., Mohyuddin, M. R., and Zhou C. Y., 2008. Few exact solutions of non-Newtonian fluid in Porous Medium with Hall effects, Journal of porous Media, vol. 11(7), pp. 669-680.
12. Zakaria, M., 2003. MHD Unsteady Free Convection Flow of a Couple Stress Fluid with one Relaxation Time Through a Porous Medium, Applied Math. & Comp., vol. 146, pp. 469-494.
13. Breugem, W. P., 2007. The Effective Viscosity of a Channel-type Porous Medium, **Physics of Fluids vol**. 19(10).
14. Batchelor, G. K., 1967. An introduction to Fluid Dynamics, Cambridge University Press London.
15. Burger, J. M., 1948. Adv. App. Mech. 1, 171-199.
16. Kambe, T. and Takao, T., 1971. J. Phys. Soc Japan 31, 591-599.
17. Kambe, T., 1983. J. Phys. Soc. Japan 52, 834-841
18. Kambe, T. and Minota, T., 1983. Proc. Roy. Soc. London A 386, 277-308.
19. Kambe, T., 1984. J. Phys. Soc. Japan 53, 13-1.
20. Kambe, T., 1986. A class of exact solutions of the Navier-Stokes equation. Fluid Dynamics research 1, 21-31.
21. Townsend, A. A., 1951. Proc. Roy. Soc. London A 208, 534-542.
22. Rose, II. A., and Sulem, P. L., 1978. J. Physique, 441-484.
23. Tshepo O Tong and Matthew T Kambule , Total stress tensors and heat fluxes of single flow through a porous viscoelastic medium**,** Life Sci J 2012; 9(1):1-12] (ISSN:1097-8135).

11/26/2012